EFFECTIVE POTENTIAL FOR EVEN-PARITY REGGE-WHEELER GRAVITATIONAL PERTURBATION EQUATIONS*

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The Schrödinger-type equation for odd-parity perturbations on a background geometry has been extended to the even-parity perturbations. This should greatly simplify the analysis for calculations of gravitational radiation from stars and from objects falling into black holes.

In examining the stability of the Schwarzschild geometry, Regge and Wheeler[1] first described a decomposition of perturbations on a spherically symmetric background geometry into tensor harmonics and wrote the perturbation equations in terms of these harmonics. There are two parity types of harmonics which have been called "odd" and "even." Regge and Wheeler found it possible to put the equations for the odd-parity harmonics into the form of a single second-order Schrödinger-type differential equation with an effective potential. Up until now, however, the even-parity equations have not been put into such a form although such a form would simplify the solution of these equations enormously. There has been considerable recent interest in these equations. Vishveshwara[2] has reexamined the stability problem for the Schwarzschild metric which Regge and Wheeler originally examined. Edelstein[3] and Zerilli[4] have looked at the problem of gravitational radiation from a particle falling in a Schwarzschild geometry, while Thorne and colleagues[5] have looked at pulsating relativistic stellar models and the gravitational radiation emitted therefrom.

We have found that the even-parity perturbation equations can also be put into the Schrödinger form described above. Using the notation of Regge and Wheeler,[1] the even-parity harmonics have coefficient functions $H_{LM}(r,\omega)$, $H_{1LM}(r,\omega)$, and $K_{LM}(r,\omega)$. H_{LM} can be given in terms of H_{1LM} and K_{LM} as described by Regge and Wheeler. H_{1LM} and K_{LM} then satisfy a coupled system of two first order linear differential equations.

There are some minor errors in the even-parity equations given by Regge and Wheeler. The correct equations have been given by Vishveshwara[2] and also the author.[4] Briefly, the functions H_{LM}, K_{LM}, and H_{1LM} satisfy the equations

$$\frac{dK_{LM}}{dr}+\frac{r-3m}{r(r-2m)}K_{LM}-\frac{1}{r}H_{LM}+\frac{1}{2}\frac{L(L+1)}{i\omega r^2}H_{1LM}=0,$$

$$\frac{dH_{LM}}{dr}+\frac{r-3m}{r(r-2m)}K_{LM}-\frac{r-4m}{r(r-2m)}H_{LM}+\left[\frac{i\omega r}{r-2m}+\frac{1}{2}\frac{L(L+1)}{i\omega r^2}\right]H_{1LM}=0,$$

$$\frac{dH_{1LM}}{dr}+\frac{i\omega r}{r-2m}K_{LM}+\frac{i\omega r}{r-2m}H_{LM}+\frac{2m}{r(r-2m)}H_{1LM}=0,$$

and also satisfy the algebraic identity

$$-\left[\frac{6m}{r}+(L-1)(L+2)\right]H_{LM}+\left[(L-1)(L+2)-\frac{2\omega^2r^3}{r-2m}+\frac{2m(r-3m)}{r(r-2m)}\right]K_{LM}+\left[2i\omega r+\frac{L(L+1)m}{i\omega r^2}\right]H_{1LM}=0.$$

The consistency of these equations is straightforwardly verified and so we may solve the algebraic identity for, say, H_{LM} and substitute this in the differential equations for H_{1LM} and K_{LM}. Let $R_{LM}=(1/\omega)H_{1LM}$. Then the equations take the form

$$dK_{LM}/dr=[\alpha_0(r)+\alpha_2(r)\omega^2]K_{LM}+[\beta_0(r)+\beta_2(r)\omega^2]R_{LM},$$

$$dR_{LM}/dr=[\gamma_0(r)+\gamma_2(r)\omega^2]K_{LM}+[\delta_0(r)+\delta_2(r)\omega^2]R_{LM}, \tag{1}$$

where α, β, γ, δ are functions only of r, L, and M, and not ω.

Now perform the transformation

$$K_{LM}=f(r)\hat{K}_{LM}+g(r)\hat{R}_{LM},\quad R_{LM}=h(r)\hat{K}_{LM}+k(r)\hat{R}_{LM},\quad dr/dr^*=n(r).$$

Then to obtain a Schrödinger-type equation we must find f, g, h, k, and n so that $d\hat{K}_{LM}/dr^*=\hat{R}_{LM}$ and

$d\hat{R}_{LM}/dr^* = [V_{LM}(r) - \omega^2]\hat{K}_{LM}$. Fortunately, the problem has a solution (many, in fact, since it turns out that one of the functions f, g, h, k, or n may be chosen arbitrarily).

If we let $n(r) = 1 - 2m/r$ where m is the mass of the Schwarzschild field we find that

$$f(r) = [\lambda(\lambda+1)r^2 + 3\lambda mr + 6m^2]/r^2(\lambda r + 3m), \quad g(r) = 1,$$

$$h(r) = i[-\lambda r^2 + 3\lambda mr + 3m^2]/(r-2m)(\lambda r + 3m), \quad k(r) = -ir^2/(r-2m), \quad \lambda = \tfrac{1}{2}(L-1)(L+2), \tag{2}$$

and that

$$r^* = r + 2m\ln(r/2m - 1), \tag{3}$$

so that

$$d\hat{K}_{LM}/dr^* = \hat{R}_{LM}, \quad d\hat{R}_{LM}/dr^* = [V_L(r^*) - \omega^2]\hat{K}_{LM}, \tag{4}$$

where

$$V_L(r) = \left(\frac{1-2m}{r}\right)\frac{2\lambda^2(\lambda+1)r^3 + 6\lambda^2 mr^2 + 18\lambda m^2 r + 18m^3}{r^3(\lambda r + 3m)^2}. \tag{5}$$

Note that the system (4) written as a single second-order equation assumes formally the Schrödinger form

$$d^2\hat{K}_{LM}/dr^{*2} + [\omega^2 - V_L(r)]\hat{K}_{LM} = 0.$$

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ac-JOSEPHSON-EFFECT DETERMINATION OF e/h WITH SUB-PART-PER-MILLION ACCURACY*

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An ac-Josephson-effect determination of e/h has been made using a new method. The accuracy of the result referred to the local voltage standard is 0.13 ppm. The accuracy referred to the National Bureau of Standards as-maintained volt is 0.46 ppm. The final value is $2e/h = 483.593\,65 \pm 0.000\,22$ MHz/$\mu V_{69\,\mathrm{NBS}}$ (0.46 ppm).

The work of Parker, Taylor, and Langenberg[1,2] has established the ac Josephson effect in systems of weakly coupled superconductors as an important factor contributing to our knowledge of the fundamental physical constants[3] and as a potential dc voltage standard.[4] Both of these applications are based upon the fact that in a Josephson-junction device biased at a dc potential difference V, there exists a supercurrent oscillating at a frequency $\nu = 2eV/h$; the frequency-voltage ratio is thus simply the fundamental physical constant $2e/h$. High-accuracy determinations of the Josephson frequency-voltage ratio have been reported by Parker et al.[2] and by Petley and Morris.[5] The former has recently been reassessed and revised by Denenstein et al.[6] Both now have quoted one-standard-deviation uncertainties of 2.2 ppm and are in excellent agreement. A significant increase in the accuracy with which the frequency-voltage ratio could be determined would have important implications for our knowledge of the fundamental physical constants and