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MODEL FOR LOW-ENERGY PROPERTIES OF THREE-NUCLEON SYSTEM*

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Calculations are presented for the properties of the three-nucleon system below the three-body breakup threshold, using simple, static, central potentials which fit the s-wave scattering lengths and effective ranges for nucleon-nucleon scattering.

In this Letter we present a number of theoretical results concerning the low-energy properties of the three-nucleon system. These results were obtained by assuming that the nucleons interact via pairwise square-well potentials in the singlet and triplet spin states; the depths and radii of the potentials were adjusted to reproduce the low-energy behavior of the (singlet and triplet) two-nucleon systems. Specifically, we calculated the triton and He^3 binding energies, the rms radii of the protons and neutrons in these nuclei, their electromagnetic charge form factors, and the low-energy doublet and quartet n-d scattering phase shifts. All of these theoretical numbers were found to be in surprisingly good agreement with the experimental values. This agreement is particularly interesting in the case of the triton binding energy and the doublet n-d scattering, both of which have been shown to be exceptionally sensitive theoretically to the details of the assumed interaction. After presenting our results we discuss their possible significance.

For all calculations we assumed equal masses for proton and neutron and took $\hbar^2/M = 41.47$ MeV fm^2. Purely central forces were used for the nucleon-nucleon interactions and in line with this we have considered only states of the three-nucleon system with total orbital angular momentum $L = 0$. The radii and potential depths of the two-body square wells were determined by fitting the latest values of the scattering lengths and effective ranges for the $^1S_0$ and $^3S_1$ n-p scattering, as given by Wilson.[1] For the singlet state we found $V_s = 14.017$ MeV, $R_s = 2.5895$ fm, which correspond to the scattering parameters $a_s = -23.714$ fm, $r_{0s} = 2.704$ fm. In the triplet state the potential parameters are $V_t = 34.406$ MeV, $R_t = 2.0719$ fm, obtained by using $a_t = 5.425$ fm, $r_{0t} = 1.749$ fm.

The three-nucleon bound-state calculations were done using the method of six-dimensional harmonics. Taking the center of mass stationary at the origin, the kinetic energy operator can be written in the form $\nabla_{\xi}^2 + \nabla_{\eta}^2$, where $\vec{\xi} = (\frac{2}{3})^{1/2} \times [\frac{1}{2}(\vec{r}_1 + \vec{r}_2) - \vec{r}_3]$, $\vec{\eta} = (\frac{1}{2})^{1/2}[\vec{r}_1 - \vec{r}_2]$ are the usual Jacobian coordinates. The orbital part of the wave function is expanded as a sum of products

of radial functions, depending on a single length coordinate $\rho = [\xi^2 + \eta^2]^{1/2}$, and a complete orthogonal set of harmonic functions depending on five angles. The harmonics are eigenfunctions of the angular part of the above Laplacian with eigenvalues $K(K+4)$, $K = 0, 1, \cdots$. They may be classified completely using the Lie algebra of SU(3) which provides exactly the five quantum numbers required. Two of the quantum numbers may be chosen to be the total orbital angular momentum L and its z component M. The functions also have convenient symmetry properties under interchange of the spatial coordinates of any two particles. These angular harmonics have been discussed by several authors; the treatments most closely related to our work are given by Dragt,[2] Zickendraht,[3] and Simonov and Badalyan.[4] The advantage of this expansion is that for reasonably soft pair potentials the three-particle wave function is very accurately represented with just the first few harmonics as labeled by the global quantum number K.

Each component of the spatial wave function is multiplied by the appropriate spin-isospin function so that the total state is completely antisymmetric. Substitution of the truncated expansion into the wave equation leads to a system of simultaneous differential equations for the radial functions. The equations are coupled by functions of ρ which are matrix elements of the pair potentials taken between the spin angular harmonics. For $L = 0$ only even values of K occur and the matrix elements for the square-well and Coulomb interactions used here may be obtained in closed form.

In the present work we used up to four harmonics. The first has $K = 0$ and is a component of the completely space-symmetric or principal S state (S_P). This contributes 98.37% to the total wave-function normalization. The next two have $K = 2$ and are states of mixed spatial symmetry (S'). One has $T = \frac{1}{2}$, the other $T = \frac{3}{2}$, and the latter is coupled only via the Coulomb potential between the two protons in He^3. The $T = \frac{3}{2}$ state contributes only 0.0013% to the normalization in He^3 and could well have been neglected entirely. The intensity of the $T = \frac{1}{2}$ S' state is 1.16% and its radial wave function is of opposite sign to the radial function of the $K = 0$ component. This is important for obtaining agreement with experiment for the Coulomb energy of He^3 and the rms charge-radius difference between H^3 and He^3, because of cross terms between the S_P and S' states in the expressions for these quantities. The last harmonic has $K = 4$ and is another component of the S_P state. Its intensity is 0.47%.

The differential equations were solved numerically to obtain the binding energies and radial functions for H^3 and He^3. The wave functions were then used to calculate the rms radii for the distribution of the like and odd particles in these nuclei, to calculate the corresponding form factors, and to check the directly calculated Coulomb energy difference by means of perturbation theory. Comparison with experiment is possible only with knowledge of the proton and neutron charge form factors. We assumed that $F_{ch}(n) = 0$ and for $F_{ch}(p)$ we took the three-pole fit of Janssens et al.[5] which gives an accurate fit to experiment over the relevant range of momentum transfer. The fit implies an rms proton radius of 0.85 fm. The electric form factors and charge radii were calculated using the formulas of Schiff.[6] The results are given in Tables I and II, which show that the calculated numbers are in good agreement with experiment except that the form factors are predicted appreciably too large for high q^2. The experimental results in Table II are those of Collard et al.[7] The discrepancy between theory and experiment for the form factors at large q^2 is probably caused by our neglect of short-range repulsions in the two body forces.

The low-energy doublet and quartet (S-wave) phase shifts for n-d scattering were obtained by solving the Fadeev equations numerically for energies below the breakup threshold. The basic procedure has been employed by many authors[8] and involves forming amplitudes of definite spin and isospin, neglecting all but s-wave two-body

Table I. Bound-state properties of three-nucleon systems.

	H^3(theor)	H^3(expt)	He^3(theor)	He^3(expt)
Binding energy (MeV)	8.72	8.49	7.97	7.73
Coulomb energy (MeV)			0.752	0.764
Rms charge radius (fm)	1.73	1.70 ± 0.05	1.85	1.87 ± 0.05
Total kinetic energy (MeV)	28.73		28.20	

Table II. Electromagnetic charge form factors for tritium and helium-3.

q^2 (fm^{-2})	$F_{ch}(H^3)$TH	$F_{ch}(H^3)$EXP	$F_{ch}(He^3)$TH	$F_{ch}(He^3)$EXP
1.0	0.624	0.622 ± 0.007	0.587	0.567 ± 0.004
1.5	0.502	0.503 ± 0.007	0.459	0.431 ± 0.004
2.0	0.408	0.387 ± 0.007	0.363	0.329 ± 0.004
2.5	0.334	0.312 ± 0.006	0.290	0.258 ± 0.003
3.0	0.276	0.267 ± 0.005	0.233	0.209 ± 0.002
3.5	0.229	0.215 ± 0.004	0.189	0.1614 ± 0.0017
4.0	0.191	0.175 ± 0.004	0.154	0.1326 ± 0.0015
4.5	0.160	0.137 ± 0.003	0.126	0.1013 ± 0.0010
5.0	0.134	0.118 ± 0.004	0.104	0.0813 ± 0.0012
6.0	0.0959	0.0758 ± 0.0041	0.0709	0.0548 ± 0.0015
8.0	0.0505	0.0295 ± 0.0039	0.0342	0.0173 ± 0.0010

interactions, and making a separable approximation to the off-shell two-body t matrix in order to reduce the problem to a finite set of coupled one-dimensional integral equations. The key step here is the introduction of the separable approximation which is facilitated in this case by a special property of the square-well potential. If we denote the partial-wave t matrix by t_l, it is a fact that for this potential the difference $t_{l+2}-t_l$ is a separable operator; this may be simply demonstrated by noting that $v_{l+2}-v_l$ is separable, where v_l is the partial-wave potential operator. This property, which is apparently unique to the square well, leads to a very natural separable approximation. That is, one observes that it is standard procedure in the (total $L=0$) Fadeev equations to neglect t_l with respect to t_0 for $l \geq 2$; this is equivalent to keeping only s-wave two-body interactions. It is thus consistent to approximate t_0 by just the separable difference between it and t_2, i.e., no new approximations are introduced by employing this separable approximation. Clearly, one can make a similar statement if one neglects all but some finite number of two-body partial waves.

Given the separable approximation it is straightforward to solve the resulting equations; the results are presented in Fig. 1 for a calculation in which both s- and d-wave two-body interactions were included. For comparison, experimental (n-d) points taken from van Oers and Seagrave[9] are also plotted. It is clear that the agreement between the calculated and experimental phase shifts is excellent; this is particularly striking for the doublet scattering, which is known to be very sensitive to the details of the two-body interaction.[10] It has been suggested that the singu-

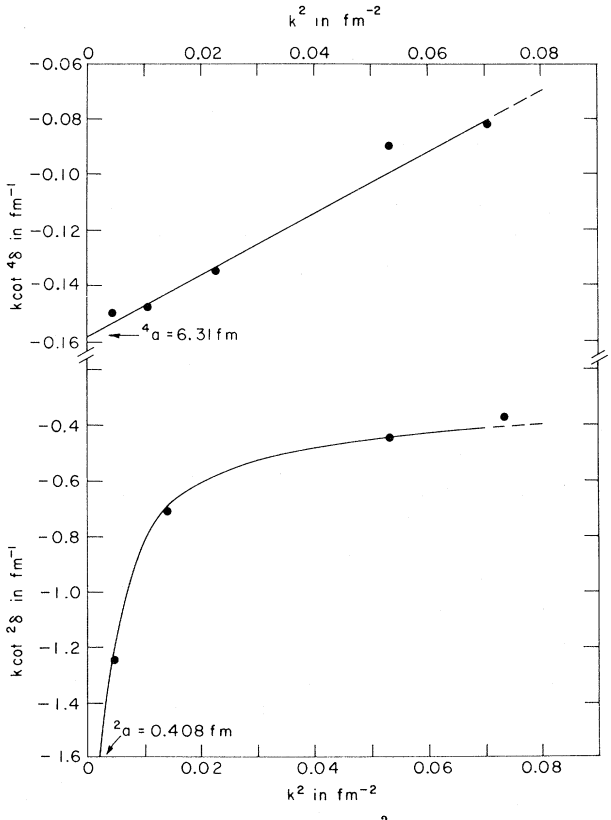

FIG. 1. Plots of $k\cot\delta$ against k^2 for the doublet and quartet n-d scattering where k is the relative wave number. Solid lines are calculated values for our square-well model, dashed lines are extensions of these curves above the breakup threshold. Solid circles are experimental values taken from van Oers and Seagrave (Ref. 9).

lar behavior of $k\cot(^2\delta)$ is due to a pole (a zero in the amplitude) just below threshold[11]; this is verified in the present model in which a pole occurs at $k^2 = -2.94\times10^{-3}$ fm^{-2}. It is worth pointing out that the agreement demonstrated in the doublet phase shift is obtained despite a relative lack of agreement with the value of the doublet scattering length quoted by van Oers and Seagrave[9] ($^2a = 0.11 \pm 0.07$ fm). Our calculated value of 0.408 fm appears to favor the previously published value of 0.7 ± 0.3 fm.[12]

The accuracy of the methods of calculation has been checked carefully by numerical studies of the rate of convergence when extra components are added to the wave functions in the harmonic method, or when further terms are included in the separable approximations to the Faddeev equations. We estimate that our quoted value for the binding energy of H^3 is within 1% of the exact result for this model. Furthermore, the results for the bound-state properties calculated with these two very different methods agree within $\frac{1}{2}$%.

While one should not conclude from this work that nature indeed employs square wells, these results have considerable significance for the general program of three-body theory. The generally accepted motivation for doing three-nucleon model calculations of this type is to probe the nuclear forces by using the three-body results to discriminate between potentials producing identical two-body behavior. Such calculations have indeed shown that a number of three-body quantities are quite sensitive to the details of the force law, and that the experimental values are not at all easy to fit simultaneously with a few parameter theory. However, if it is possible to correlate important features of the two- and three-body low-energy data with an apparently nonrealistic potential, as in the case of the square well, one has to question seriously the nature of the results one can expect from this program.

On the other hand, it may be that the square well has something to tell us about the mathematical properties of the "real" potential. It is probably worth noting in this respect that the off-shell t-matrix elements generated by this potential undergo numerous changes of sign, a feature not shared by other simple potentials such as the Yukawa and exponential which do not give good simultaneous agreement with experiment for the two- and three-body data. It has recently been pointed out to us[13] that a somewhat more "realistic" nucleon potential, the Reid potential, also possesses this oscillatory off-shell property. In fact, for the case of the $^1S_0$ potential, which is the only one for which we have comparisons available, the Reid potential and our simple square well give closely similar values for the phase shift up to a relative energy of 80 MeV. Furthermore, the half-off-shell t-matrices generated by the two potentials have closely matching values for a wide range of off-shell momentum.

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