DOES RENORMALIZED PERTURBATION THEORY DIVERGE?

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Renormalized perturbation theory in the anharmonic oscillator diverges.

In many field-theoretic models,[1] and in particular in the anharmonic oscillator,[2,3] unrenormalized perturbation theory diverges. It is thus interesting to ask whether reexpressing a field theory in terms of its physical masses and coupling constants (renormalizing) could give rise to a convergent perturbation theory. In this paper we show that for the anharmonic oscillator model the answer to this question is no. Specifically, we show that the energy levels of the anharmonic oscillator, when expressed terms of the renormalized coupling constant Λ and mass M, cannot be analytic functions of Λ near $\Lambda=0$ for fixed M regardless of the definition of Λ. We conclude with an illustrative example using a particular choice for Λ.

Our result that the renormalized perturbation series diverges may help to clear up some of the mystique surrounding the renormalization process. At least in the anharmonic oscillator, renormalizing does exactly what it is supposed to do – it replaces "bare" parameters with physical parameters – and nothing more.[4] Furthermore, our result strongly suggests that renormalized perturbation theory also diverges in more realistic field theories.

The anharmonic oscillator is an ideal field-theoretic model in which to study perturbation theory for several reasons:

(a) The model is simple enough to allow a completely general approach to renormalization. That is, although the renormalized coupling constant Λ is not unique, our analysis holds for all choices of Λ.

(b) The model is complicated enough to exhibit a nontrivial and divergent unrenormalized Feynman perturbation series for all n-point Green's functions. In fact, the anharmonic oscillator is a φ^4 field theory in one-dimensional space-time and its diagrammatic expansion is identical to that of the usual four-dimensional φ^4 theory.[2,3]

(c) The perturbation series is finite in every order.[2,3] Thus, our analysis is not complicated by the ultraviolet divergences that appear in higher-dimensional space-time.

Using the notation of Ref. 2, the unrenormalized anharmonic oscillator Hamiltonian is

$$H=\tfrac{1}{2}\dot{\varphi}^2+\tfrac{1}{2}m^2\varphi^2+\lambda\varphi^4, \tag{1}$$

where $[\varphi,\dot{\varphi}]=i$, and m and λ are the bare mass and coupling constant. The "physical" quantities of the theory are $E_n(m,\lambda)$, $n=0, 1, 2, \cdots$, the energy levels of H. The E_n take the form $E_n(m,\lambda)=mF_n(\omega)$ where $\omega=\lambda m^{-3}$.

To renormalize the theory we first rescale the energy levels so that the ground-state (vacuum) energy is 0. Thus we define a new Hamiltonian.

$$\mathcal{H}\equiv H-E_0, \tag{2}$$

whose energy levels are $\mathcal{E}_n=E_n-E_0$.

Then we rewrite $\mathcal{E}_n$ in terms of the mass of the physical one-particle state $M=\mathcal{E}_1$ and the renormalized coupling constant Λ:

$$\mathcal{E}_n=\mathcal{E}_n(M,\Lambda)=M\mathfrak{F}_n(\Omega), \tag{3}$$

where

$$\Omega=\Lambda M^{-3}. \tag{4}$$

Because Ω is proportional to the renormalized coupling constant, as Ω approaches zero along the real axis in the Ω plane, the energy levels must assume their free (harmonic-oscillator) values, $\mathfrak{F}_n(0)=n$. Ω is not required to exhibit any further properties and is certainly not uniquely defined.

We will establish the divergence of renormalized perturbation theory by showing that the assumption that each $\mathfrak{F}_n(\Omega)$ is an analytic function of Ω within a circle of radius R_n about the origin leads to a contradiction. This result follows directly from three crucial qualitative properties of the eigenvalues of the anharmonic oscillator when expressed in terms of the bare parameters m and λ. Each of these properties is ascertained using the WKB techniques discussed in Refs. 2 and 3. We list and discuss these properties below:

(i) $F_n(\omega)$ has at least one infinite sequence of singularities in the ω plane which are all distinct from those of $F_p(\omega)$, $n\neq p$. More specifically, for $n\geq 2$, $F_n(\omega)$ has two infinite sequences of singularities in the ω plane near $\arg\omega=3\pi/2$ with a limit point at $\omega=0$. We denote these singularities by $\omega_{n,n+2}{}^i$ and $\omega_{n-2,n}{}^i$. For $n=0$ and $n=1$ only the sequences $\omega_{0,2}{}^i$ and $\omega_{1,3}{}^i$ occur. An explicit formula and graph showing the locations of these singularities in the ω plane may be found in Ref. 3.

(ii) $F_n'(\omega_{p,p+2}{}^i)$, where $F_n'\equiv dF_n/d\omega$, is infinite when $p=n$ or $n-2$ and is finite otherwise. $\omega_{p,p+2}{}^i$ are square-root-type branch points. (F_p and F_{p+2} undergo level crossing at $\omega_{p,p+2}{}^i$. For a graphical picture of level crossing see Ref. 3.) Moreover, near $\omega_{p,p+2}{}^i$, the square-root behavior of F_p must be

$$F_p(\omega)\sim a+b(\omega-\omega_{p,p+2}{}^i)^{1/2}+\cdots,\quad b\neq 0. \tag{5}$$

To verify Eq. (5) one uses the approximate implicit relation between $F(\omega)$ and ω for ω near the branch points given by WKB theory[3]:

$$f(F,\omega)=\frac{\Gamma(\frac{1}{4}+\frac{1}{2}F)}{\Gamma(\frac{1}{4}-\frac{1}{2}F)}-\exp\left[\frac{5\pi i}{4}-F\log(\tfrac{1}{2}\rho)+\frac{i}{3\rho}\right]=0, \tag{6}$$

where $\rho=\omega e^{-3\pi i/2}$. Using Eq. (6) we calculate $dF(\omega)/d\omega=(\partial f/\partial\omega)(\partial f/\partial F)^{-1}$ and observe that $\partial f/\partial F$ is zero at the branch points while $\partial f/\partial\omega$ is not. This establishes Eq. (5).

(iii) If $n<p$, then $F_n(\omega_{p,p+2}{}^i)\to n+\frac{1}{2}$ as $i\to\infty$. This statement is verified by substituting the formula[2,3] giving the branch points in the ω plane into the implicit relation between $F(\omega)$ and ω in Eq. (6). Letting $i\to\infty$, one observes that F_n approaches its harmonic oscillator value.

We now show that a contradiction arises from the assumption that the energy levels are analytic functions of Ω near the origin. We observe that we can associate a sequence $\Omega_{n,n+2}{}^i=\Omega(\omega_{n,n+2}{}^i)$ with $\omega_{n,n+2}{}^i$ via the relation

$$\mathfrak{F}_p(\Omega)=\frac{F_p(\omega)-F_0(\omega)}{F_1(\omega)-F_0(\omega)}. \tag{7}$$

The $\Omega_{n,n+2}{}^i$ have the origin in the Ω plane as their limit point. To prove this, we note that by property (iii), the right-hand side of Eq. (7) approaches p as $i\to\infty$. But $\mathfrak{F}_p(\Omega)$ is assumed analytic in Ω near $\Omega=0$ with $\mathfrak{F}_p(0)=p$. Hence, near $\Omega=0$, $\mathfrak{F}_p(\Omega)$ behaves like $p+\alpha\Omega^q$ with q a positive integer. Thus, as $i\to\infty$, $\Omega_{n,n+2}{}^i$ must approach 0 like

$$\alpha^{-1/q}\left\{\frac{F(\omega_{n,n+2}{}^i)-F_0(\omega_{n,n+2}{}^i)}{F_1(\omega_{n,n+2}{}^i)-F_0(\omega_{n,n+2}{}^i)}-p\right\}^{1/q}.$$

Next, we invoke the chain rule for differentiation:

$$\frac{\partial\mathfrak{F}_n(\Omega)}{\partial\omega}=\frac{\partial\mathfrak{F}_n}{\partial\Omega}\frac{\partial\Omega}{\partial\omega}. \tag{8}$$

Note that because of property (ii) and Eq. (7) the left-hand side of Eq. (8) is infinite at $\omega=\omega_{n,n+2}{}^i$. But we have established that it is possible to choose an i_0 sufficiently large that when $i>i_0$ the associated $\Omega_{n,n+2}{}^i$ lie within the assumed region of analyticity $|\Omega|<R_n$ of $\mathfrak{F}_n(\Omega)$. Thus $\partial\mathfrak{F}_n/\partial\Omega$ is finite and $\partial\Omega/\partial\omega$ is infinite at points $\omega_{n,n+2}{}^i$, $i>i_0$. By the same argument, $\partial\Omega/\partial\omega$ is infinite at points $\omega_{p,p+2}{}^j$, $j>j_0$, $p\neq n$, and $p\neq n\pm 2$. But, by property (i), $\partial\mathfrak{F}_n/\partial\omega$ is finite at such points $\omega_{p,p+2}{}^j$. Hence, $\partial\mathfrak{F}_n/\partial\Omega$ must vanish at the associated sequence of points $\Omega_{p,p+2}{}^j$, $j>j_0$.

However, an analytic function which vanishes on a sequence of points with a limit point must vanish identically. Thus $\mathfrak{F}_n(\Omega)$ must be a constant,

$$\mathfrak{F}_n(\Omega)=n, \tag{9}$$

and the energy levels take on their noninteracting values. This establishes the contradiction.

We have shown that it is not possible to define any renormalized coupling constant Ω which provides an analytic continuation from the free (harmonic oscillator) energy levels to those of the interacting theory.

We conclude this note with a concrete example which illustrates for a particular choice of Ω how the predicted nonanalyticity appears. One reasonable choice for the renormalized coupling constant Ω is

$$\Omega(\omega)=\frac{F_2(\omega)-2F_1(\omega)+F_0(\omega)}{F_1(\omega)-F_0(\omega)}. \tag{10}$$

Note that $\Omega\to 0$ as $\omega\to 0$. To show that $\mathfrak{F}_n(\Omega)$ for $n>2$ is not an analytic function of Ω for Ω small, consider a branch point of $F_n(\omega)$ in the ω plane, $\omega_{n,n+2}{}^i$. Property (i) allows us to take a neighborhood N of $\omega_{n,n+2}{}^i$ so small that it includes no other branch points. Thus, F_0, F_1, and F_2 are all analytic in the neighborhood N. Now consider a closed path in N which contains $\omega_{n,n+2}{}^i$. Equation (10) defines the analytic image of this path in the Ω plane as a closed path about $\Omega_{n,n+2}{}^i$. Hence, $\Omega_{n,n+2}{}^i$ is a branch point of $\mathfrak{F}_n(\Omega)$. Finally, Eq. (10) and property (iii) imply that as $\omega_{n,n+2}{}^{l}\to 0$, $\Omega_{n,n+2}{}^{l}\to 0$. Hence, $\mathfrak{F}_n(\Omega)$ cannot be analytic about the origin. In fact, the origin is a nonisolated singularity of the energy eigenvalues.

The nonanalyticity we have observed in the above example was easy to establish because Ω was given as an analytic function of a finite number of energy levels. The general argument includes the possibility that Ω is not simply expressible in terms of the energy levels.

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[1]For example, for self-interacting Bose fields in two-dimensional space-time see A. M. Jaffe, Commun. Math. Phys. 1, 127 (1965).

[2]C. M. Bender and T. T. Wu, Phys. Rev. Letters 21, 406 (1968).

[3]C. M. Bender and T. T. Wu, Phys. Rev. 184, 1231 (1969).

[4]Renormalization is not an answer to the problem of extracting physical information from divergent series. One must either take the coupling constant very small and use an asymptotic approximation or else introduce summability methods when this is not possible. Padé techniques have been used to sum the perturbation expansion in the anharmonic oscillator. For a discussion of these techniques and a verification of some of the properties of the anharmonic oscillator discovered in Refs. 2 and 3 see B. Simon, "Coupling Constant Analyticity for the Anharmonic Oscillator" (to be published), and J. J. Loeffel, A. Martin, B. Simon, and A. S. Wightman, "Padé Approximants and the Anharmonic Oscillator" (to be published).

PROTON-PROTON ELASTIC SCATTERING AT 15, 20, AND 30 BeV/c *

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Proton-proton elastic scattering has been measured at 15.2, 20.0, and 29.7 BeV/c. The data at 20.0 BeV/c confirm the presence of a break at $-t\cong 1.2$ (BeV/c)^2. The data at 29.7 BeV/c show essentially the same behavior. The cross section is still falling with increasing energy in this $-t$ range.

In the course of an extensive wire-plane experiment at the Brookhaven alternating-gradient synchrotron (AGS), our group accumulated data on elastic and inelastic p-p scattering at a number of angles and energies. The inelastic-scattering data along with details of the apparatus and the technique have already been reported.[1] A distinctive feature of these data is the presence of sharp breaks in the cross sections for single isobar production near $-t=1$ (BeV/c)^2. Recent proton-proton elastic-scattering measurements[2] near 20 BeV/c also exhibit a pronounced break at $-t\simeq 1.2$ (BeV/c)^2. Many models have suggested that a dip or break might occur in this region. Some of these, based on the optical model proposed by Yang and his collaborators,[3] give an asymptotic form which is related to the proton electromagnetic form factor. Others, such as the Regge-pole model of Frautschi and Margolis,[4] are able to predict the energy dependence of the cross section. To distinguish between the predictions, it is important to obtain proton elastic-scattering data at as high an energy as possible in order to observe how the break changes with energy. This Letter presents elastic cross-section measurements at 15.2, 20.0, and 29.7 BeV/c, making it possible to compare directly with the data of Allaby et al.[2] and also to observe the change of the cross section with increasing energy.

The cross sections were measured by detecting the high-energy proton with a magnetic spectrometer which utilized wire planes connected "on-line" to a PDP-6 computer. A beam of 10^5 to 10^8 protons per pulse was obtained by diffraction scattering at one degree from an internal