energy difference in the Hartree approximation and also assume that $|\alpha\rangle$ and $|a\rangle$ have the same spatial wave functions. Then the symmetry energy ($U_{\alpha a}$ say, a positive quantity) is given by $\epsilon_\alpha{}^p - \epsilon_a{}^n = -u_{\alpha a} \approx -(2T_0)^{1/2}\mathcal{V}_{\alpha,a\lambda}$ on comparison with the second member of Eq. (15). Equation (14b) now reduces to a 2×2 determinantal equation for the roots E_i. To leading order in $(2T_0)^{-1} \ll 1$ one has $E_A - \epsilon_a{}^n \approx u_{\alpha a}/2T_0$ and $E_{AA} - \epsilon_a{}^n \approx -U_{\alpha a} - U_{\alpha a}/2T_0$ with renormalization coefficients $Z_A \approx (2T_0+1)^{-1}$ and $Z_{AA} \approx 2T_0(2T_0+1)^{-1}$, respectively. It is clear that under these approximations the quasiparticle states at energies E_A and E_{AA} correspond to the analog and antianalog states of the Lane model, containing, respectively, Z_A and Z_{AA} of the single-particle strength, and therefore that these states have good isospin $(T_0+\frac{1}{2}, T_0-\frac{1}{2})$ and $(T_0-\frac{1}{2}, T_0-\frac{1}{2})$.

The ingredients of our discussion—an exact symmetry of the nuclear Hamiltonian, its breaking by the HF vacuum, and the occurrence of a zero-energy analog excitation—are strongly reminiscent of the situation encountered in relativistic field theory[7] and in nonrelativistic many-body problems.[8] The precise relationship, if any, between the analog excitation and the massless guage fields and Goldstone bosons is being investigated.

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SOLUTIONS OF TWO PROBLEMS IN THE THEORY OF GRAVITATIONAL RADIATION*

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The evolution of an elongated rotating configuration by gravitational radiation and the possibility of a secular instability being induced by it are considered in the context of the classical homogeneous figures of Maclaurin and Jacobi. The triaxial Jacobian ellipsoid evolves in the direction of increasing angular velocity and approaches (exponentially) the point of bifurcation where it ceases to radiate. Further, radiation reaction does not make the Maclaurin spheroid secularly unstable past the point of bifurcation.

In a recent paper,[1] the equations of hydrodynamics governing a perfect fluid have been derived, consistently with Einstein's field equations of general relativity, to sufficient approximation that the terms representing the reaction of the fluid to the emission of gravitational radiation are explicitly present [see Eq. (8) below]. With the aid of these equations, two problems (which have arisen in current discussions of the pulsars[2-4]) in the theory of gravitational radiation can be solved. In this Letter, we formulate these problems, present their solutions in a simple realizable context, and conclude with some brief comments on the bearing of the results derived for the problem of gravitational collapse.

The formulation of the problems.—The first problem concerns the evolution of a rotating fluid mass emitting gravitational radiation. The second problem concerns the possibility that the dissipation of energy by gravitational radiation induces "secular instability" in the manner that viscosity sometimes does.

We shall consider the foregoing two problems in the context of the classical theory of uniformly rotating homogeneous fluid masses.[5] In this theory it is known that along the sequence of axially symmetric configurations – the Maclaurin spheroids – a point of bifurcation occurs where a new sequence of triaxial ellipsoids – the Jacobi ellipsoids – branches off; and further, that the Maclaurin spheroid becomes secularly unstable at the point of bifurcation (cf. E.F.E., §37).

The Maclaurin spheroid, because of its axial symmetry, cannot radiate gravitational waves, but the Jacobi ellipsoids can and will. The first of the two problems, in this context, concerns the consequent evolution of the Jacobi ellipsoid. The second problem concerns the secular stability of the Maclaurin spheroid at the point of bifurcation when allowance is made for the dissipation of energy by gravitational radiation during its oscillations.

The evolution of the Jacobi ellipsoid by gravitational radiation. – The structure of the Jacobi ellipsoid is governed by two equations (E.F.E., §39): the equation

$$a_1^2 a_2^2 A_{12} = a_3^2 A_3, \tag{1}$$

which determines the geometry (i.e., the ratio of the axes a_1, a_2, and a_3) of the ellipsoid, and the equation

$$\Omega^2/\pi G\rho = 2B_{12}, \tag{2}$$

which determines the angular velocity Ω that is to be associated with each Jacobian figure. In Eqs. (1) and (2), $A_{ij\ldots}$ and $B_{ij\ldots}$ are certain definite integrals defined by (E.F.E., §21)

$$A_{ij\ldots} = a_1 a_2 a_3 \int_0^\infty \frac{du}{\Delta(a_i^2+u)(a_j^2+u)\cdots},$$

$$B_{ij\ldots} = a_1 a_2 a_3 \int_0^\infty \frac{u\,du}{\Delta(a_i^2+u)(a_j^2+u)\cdots},$$

and

$$\Delta^2 = (a_1^2+u)(a_2^2+u)(a_3^2+u). \tag{3}$$

Also, in Eq. (2), ρ is the assumed constant density of the configuration and G is the constant of gravitation.

The evolution of the Jacobi ellipsoid as it radiates gravitationally can be determined quite simply without an explicit appeal to the full equations of motion. Thus, in the standard linearized theory of gravitational radiation (with which the detailed theory of Ref. 1 is consistent), the rate of radiation of the angular momentum L by a mass, rotating uniformly with an angular velocity Ω and whose principal components of the moment-of-inertia tensor in the equatorial plane are I_{11} and I_{22}, is given by

$$dL/dt = -(32G/5c^5)(I_{11}-I_{22})^2\Omega^5. \tag{4}$$

For the Jacobi ellipsoid this equation gives

$$d[(a_1^2+a_3^2)\Omega]/dt = -(32GM/25c^5)(a_1^2-a_2^2)^2\Omega^5, \tag{5}$$

where M denotes the mass of the ellipsoid.[6] On the assumption that the emission of gravitational radiation alters the figure and the speed of rotation of the ellipsoid at a rate that is slow compared with its instantaneous rate of rotation – an assumption that is fully justified under the circumstances in which Eq. (4) is applicable – the evolution of the ellipsoid is uniquely determined by Eq. (5) and the requirements that Eqs. (1) and (2) continue to specify the figure and the angular velocity at each instant and that $a_1a_2a_3$ remain constant (on account of the assumed homogeneity of the configuration).

The result of integrating Eq. (5), together with the subsidiary conditions provided by Eqs. (1) and (2) and $a_1a_2a_3$ = constant, is exhibited in Fig. 1. For the sake of definiteness it has been supposed that the initial Jacobi ellipsoid (at $t=0$) is the one that is the most elongated compatible with stability, namely, $a_2/a_1 = 0.4322$ and $a_3/a_1 = 0.3451$, where $\Omega^2/\pi G\rho = 0.2840$ (E.F.E., p. 110).

It will be observed that the angular velocity increases during the evolution of the Jacobi ellipsoid. This result is indeed to be expected since along the Jacobian sequence the angular velocity decreases while the angular momentum increases (see Figs. 5 and 6 in E.F.E., p. 79). An inference

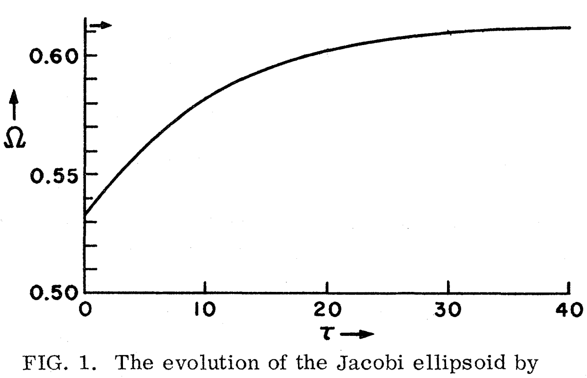

FIG. 1. The evolution of the Jacobi ellipsoid by gravitational radiation. The ordinate measures the angular velocity of rotation Ω in the unit $\pi G\rho^{1/2}$, and the abscissa the time τ in the unit T defined in Eq. (7). The arrow indicates the angular velocity which the ellipsoid attains as it approaches, asymptotically, a spheroidal nonradiating state.

of some general importance in this connection is that the energy dissipated by the emission of gravitational radiation need not necessarily be at the expense of the rotational kinetic energy only: It could equally derive from the potential and/or the internal energy.

From Eqs. (1), (2), and (5) it can be shown that the Jacobi ellipsoid approaches the nonradiating Maclaurin spheroid, at the point of bifurcation, asymptotically in the manner

$$(1-a_2/a_1)\rightarrow \text{constant}\times e^{-\tau/16.039} \tag{6a}$$

and

$$(\Omega_{\mathrm{Mc}}{}^*-\Omega)\rightarrow \text{constant}\times e^{-2\tau/16.039}, \tag{6b}$$

where $\Omega_{\mathrm{Mc}}{}^*=0.6117(\pi G\rho)^{1/2}$ denotes the angular velocity of the limiting Maclaurin spheroid and τ is the time measured in the unit

$$T=(25/18)\,(\bar{a}/R_{\mathrm{S}})^3(\bar{a}/c). \tag{7}$$

where $\bar{a}=(a_1a_2a_3)^{1/3}$ is the constant mean radius of the ellipsoid, c is the velocity of light, and $R_{\mathrm{S}}=2GM/c^2$ is the Schwarzschild radius.

It may be noted here that for $\bar{a}=20$ km and $M=1$ solar mass, $T=2.90\times10^{-2}$ sec so that the time constant in Eq. (6b), namely, $8.02T$, is 0.232 sec in contrast to the period of rotation, 2.92×10^{-3} sec at the point of bifurcation. The assumption that the Jacobi ellipsoid evolves "adiabatically" is thus amply justified under these circumstances.

The secular stability of the Maclaurin spheroid at the point of bifurcation. – The effect of radiation reaction on the stability of the Maclaurin spheroid cannot be determined without an explicit knowledge of the equations of motion including the terms representing this reaction. In a frame of reference in which the center of mass is at rest, these terms are [Ref. 1, Eq. (101)]

$$\frac{1}{c^5}\left[-\rho Q_{00}{}^{(5)}\frac{dv_\alpha}{dt}-\frac{1}{2}\rho v_\alpha\frac{dQ_{00}{}^{(5)}}{dt}-\rho\frac{d}{dt}(v_\mu Q_{\mu\alpha}{}^{(5)})-\frac{1}{2}\rho Q_{\mu\nu}{}^{(5)}\frac{\partial\mathfrak{V}_{\mu\nu}}{\partial x_\alpha}+\frac{1}{5}\rho x_\alpha G\frac{d^5I_{\mu\mu}}{dt^5}-\frac{3}{5}\rho x_\mu G\frac{d^5I_{\mu\alpha}}{dt^5}\right], \tag{8}$$

where

$$Q_{00}{}^{(5)}=\frac{4}{3}G\frac{d^3I_{\mu\mu}}{dt^3},\quad Q_{\alpha\beta}{}^{(5)}=2G\frac{d^3I_{\alpha\beta}}{dt^3}-\frac{2}{3}G\delta_{\alpha\beta}\frac{d^3I_{\mu\mu}}{dt^3},$$

and

$$\mathfrak{V}_{\alpha\beta}(\vec{x})=G\int_V\rho(\vec{x}')\frac{(x_\alpha-x_\alpha')(x_\beta-x_\beta')}{|\vec{x}-\vec{x}'|^3}d\vec{x}'. \tag{9}$$

In Eqs. (8) and (9), $I_{\alpha\beta}$ denotes the moment-of-inertia tensor and v_α the velocity component. Equations (8) and (9) are written in the notation of Cartesian tensors with the summation convention over repeated indices; the Greek indices refer to the spatial coordinates and run through 1, 2, and 3.

In treating the stability of the Maclaurin spheroid, we must first transform the terms (8) to a rotating frame of reference. In view of the high order of the time derivatives that occur in these terms, the required transformation is somewhat complicated; but we shall not give any of the details here.

Since the terms (8) vanish for the unperturbed Maclaurin spheroid, it will suffice to evaluate them for the perturbed configuration. Describing the perturbation (in a uniformly rotating frame) in terms of a Lagrangian displacement of the form $\vec{\xi}(\vec{x})e^{\lambda t}$, where λ is a characteristic-value parameter to be determined, we can treat the present problem exactly by the "virial method" described in E.F.E., Chaps. 2 and 5. In this method the relevant perturbation equations are expressed in terms of the quantity

$$V_{\alpha\beta}=\int_V\rho(\xi_\alpha x_\beta+\xi_\beta x_\alpha)d\vec{x}. \tag{10}$$

For determining the effect of the radiation reaction on the stability of the Maclaurin spheroid, it will suffice to consider the mode of oscillation which becomes neutral at the point of bifurcation and dynamically unstable further along the sequence. For this mode of oscillation $V_{\mu\mu}=0$ [cf. E.F.E., §33(b)]; and the procedure outlined in E.F.E., §§33 and 39, suitably adapted to this problem, leads to the pair of equations

$$[\lambda^2+2(2B_{11}-\Omega^2)+2DQ_1]V_{12}+(\lambda\Omega-\tfrac{1}{2}DQ_2)(V_{11}-V_{22})=0, \tag{11}$$

$$[\lambda^2+2(2B_{11}-\Omega^2)+2DQ_1](V_{11}-V_{22})-4(\lambda\Omega-\tfrac{1}{2}DQ_2)V_{12}=0, \tag{12}$$

where λ^2 and Ω^2 are measured in the unit $\pi G\rho$,

$$D=GI_{11}(\pi G\rho)^{3/2}c^{-5} \tag{13}$$

is a dimensionless constant, and

$$Q_1=2\lambda(\lambda^2-12\Omega^2)(\Omega^2-2B_{11})-\tfrac{3}{5}\lambda^5+8\lambda^3\Omega^2+16\lambda\Omega^4,$$
$$Q_2=-8\Omega(3\lambda^2-4\Omega^2)(\Omega^2-2B_{11})+8\lambda^4\Omega-(128/5)\Omega^5. \tag{14}$$

Equations (11) and (12) lead to the characteristic equation

$$\sigma^2-2\sigma\Omega-2(2B_{11}-\Omega^2)+4iD(2\Omega-\sigma)^3[(2B_{11}-\Omega^2)+\tfrac{1}{10}(2\Omega-\sigma)(4\Omega+3\sigma)]=0, \tag{15}$$

where we have written $i\sigma$ in place of λ. Letting

$$\sigma=\sigma_0+\Delta\sigma, \tag{16}$$

where σ_0 is the characteristic frequency in the absence of radiation reaction, we find that to first order in D,

$$\Delta\sigma=-iD[2(2\Omega-\sigma_0)^3/(\sigma_0-\Omega)][(2B_{11}-\Omega^2)+\tfrac{1}{10}(2\Omega-\sigma_0)(4\Omega+3\sigma_0)]. \tag{17}$$

By making use of the equation satisfied by σ_0, equation (17) can be brought to the simple form

$$\Delta\sigma=-\tfrac{2}{5}iD(2\Omega-\sigma_0)^5/(\sigma_0-\Omega). \tag{18}$$

In the absence of radiation reaction, Eq. (15) allows two roots:

$$\sigma_0^{(1)}=\Omega-(4B_{11}-\Omega^2)^{1/2}$$

and

$$\sigma_0^{(2)}=\Omega+(4B_{11}-\Omega^2)^{1/2} \tag{19}$$

It is known (E.F.E., p. 99) that viscosity induces a secular instability of the mode $\sigma_0^{(1)}$ which becomes neutral at the point of bifurcation where $\Omega^2=2B_{11}$. We shall now show that radiation reaction induces a similar instability of the mode $\sigma_0^{(2)}$ which acquires a frequency 2Ω at the same point.

For the mode $\sigma_0^{(2)}$ Eq. (18) gives[7]

$$i\Delta\sigma=\tfrac{2}{5}D[(4B_{11}-\Omega^2)^{1/2}-\Omega]^5/(4B_{11}-\Omega^2)^{1/2}, \tag{20}$$

while the corresponding equation, giving the effect of a kinematic viscosity (ν) of the fluid on the mode $\sigma_0^{(1)}$, is

$$i\Delta\sigma=-\frac{5\nu}{a_1^2}\frac{(4B_{11}-\Omega^2)^{1/2}-\Omega}{(4B_{11}-\Omega^2)^{1/2}}. \tag{21}$$

From Eq. (20) it follows that while the mode $\sigma_0^{(2)}$ is damped by gravitational radiation prior to the point of bifurcation at $\Omega^2=2B_{11}$, it is amplified in the interval $4B_{11}>\Omega^2>2B_{11}$. Thus radiation reaction, like viscosity, makes the Maclaurin spheroid unstable beyond the point of bifurcation; but the mode that is made unstable by radiation reaction is not the same one that is made unstable by viscosity.

Some general comments. – The foregoing solutions to two definite problems in the theory of gravitational radiation suggest that the following considerations may be relevant to the theory of gravitational collapse following a supernova outburst.

A rapidly rotating highly condensed configuration may, in the first instance, form as a result of the collapse; and it is not unlikely that the rotating configuration may, in fact, be similar to a Jacobian ellipsoid at the limit of its stability. Then by gravitational radiation, its angular velocity will increase (in the manner illustrated in Fig. 1) and the object will approach a point of bifurcation where the object becomes spheroidal and nonradiating. But once it reaches the point of bifurcation, radiation reaction will make the configuration secularly unstable; and it is possible that further development may proceed in the direction of fragmentation. In any event, the fact that radiation reaction can induce secular instabilities must have an important bearing on what may happen during the late stages of gravitational collapse.

I am indebted to Miss Donna Elbert for carrying out the integration of Eq. (5) exhibited in Fig. 1.

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[5]For an account of this theory see S. Chandrasekhar, Ellipsoidal Figures of Equilibrium (Yale University, New Haven, 1969). This book will be referred to as E.F.E.

[6]It should be noted that $d(a_1^2+a_2^2)/dt$ and $d\Omega/dt$ both behave like $(a_1-a_2)^2$ as one approaches the point of bifurcation.

[7]The singularity which this expression manifests at $\Omega^2=4B_{11}$ (where dynamical instability sets in) is spurious: Eq. (15), without the substitution (16), leads to the result $\sigma=\Omega\pm(0.4D\Omega^5)^{1/2}(1+i)$ at this point.

EVIDENCE FOR THE EXISTENCE OF A Z^*†

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Results of a phase-shift analysis with and without the use of a Regge-pole model in K^+p elastic scattering from 0.86 to 1.95 GeV/c are described. We have obtained four possible solutions, all of which indicate resonantlike behavior in one partial-wave amplitude. Three of our solutions yield a $P_{3/2}$ partial wave with a behavior consistent with the Breit-Wigner resonance formula. The properties of such a Z^* resonance are described.

We have reported earlier a preliminary phase-shift analysis of K^+p elastic-scattering data from 0.86 to 1.95 GeV/c.[1] Since then an extensive random search for solutions has been carried out. The data used in the analysis include new polarization measurements[2] at 1.89 GeV/c which we obtained using a butanol-alcohol target[3] at 26 kG. These new measurements are in good agreement with the old ones which were taken with a lanthanum-magnesium-nitrate target. We have also included recent polarization data at 0.86 and 0.96 GeV/c by Andersson et al.[1] We have obtained four possible solutions, all of which indicate resonantlike behavior in one partial-wave amplitude. The same results were obtained when analysis was repeated using predictions for high partial waves from a Regge-pole model.

In our energy-independent analysis, we have found typically about 50 solutions ($\chi^2/N_{\rm DF}\leq 2.5$) at each momentum. By examining all the partial waves simultaneously, we noted that these solutions clustered into several areas on Argand diagrams. To select the best of these, we first required that each group vary with increasing momentum on the Argand diagram in a smooth, continuous way in each partial wave; this test elimi-

Table I. Outline of phase-shift solutions.

Momentum (GeV/c)	$l_{\rm max}$	No. of parameters	No. of data points	Values of χ^2 I	II	III	IV
0.86	2	10	44	49	84	84	63
0.96	2	10	42	43	57	56	57
1.21	2	10	45	27	50	50	27
1.37	3	14	64	85	111	111	97
1.45	3	14	82	68	74	76	86
1.70	3	14	59	30	30	29	29
1.95	3	14	110	134	134	134	135