

elements, the search for which constitutes a separate work and will not be described here.

Experiments are in progress to determine what, if any, changes occur in d in the dissipative region of film flow. Results from these will be reported elsewhere along with a fuller account of the present investigation.

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¹See, for example, the review of L. C. Jackson and L. G. Grimes, *Advan. Phys.* **7**, 435 (1958), and references therein.

²P. W. F. Gribbon and L. C. Jackson, *Can. J. Phys.* **41**, 1047 (1963).

³K. A. Pickar and K. R. Atkins, *Phys. Rev.* **178**, 389 (1969).

⁴V. M. Kontorovich, *Zh. Eksperim. i Teor. Fiz.* **30**, 805 (1956) [*Soviet Phys. JETP* **3**, 770 (1956)]. See also, R. G. Arkhipov, *Zh. Eksperim. i Teor. Fiz.* **33**, 16 (1957) [*Soviet Phys. JETP* **6**, 90 (1958)].

⁵Emerson and Cummings, Inc., Gardens, Calif.

⁶W. E. Keller and E. F. Hammel, *Phys. Rev. Letters* **17**, 998 (1966); L. J. Campbell, E. F. Hammel, D. M. Jones, and W. E. Keller, in *Proceedings of the Eleventh International Conference on Low Temperature Physics, St. Andrews, Scotland, 1968*, edited by J. F. Allen, D. M. Finlayson, and D. M. McCall (St. Andrews University, St. Andrews, Scotland, 1969), Vol. 1, p. 182.

⁷See, for example, R. Bowers, *Phil. Mag.* **44**, 485 (1953).

⁸All estimates of d take into account that film covers both plates of the capacitor and assume that only atoms in the gap defined by the inner plate contribute to the observed capacitance.

⁹As calculated from the cell geometry at room temperature and the atomic polarizability of He, $\alpha = 1.950 \times 10^{22}$ atoms/cm³ pF.

¹⁰I. E. Dzyaloshinskii, E. M. Lifshitz, and L. P. Pitaevskii, *Advan. Phys.* **10**, 165 (1961).

POSSIBLE FLOW-DEPENDENT TEMPERATURE GRADIENTS IN He FILMS

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Various experiments, especially recent experiments by Keller, clearly demonstrate that the thickness of a helium film is independent of the flow rate Q . This is consistent with our current picture of the hydrodynamics of a helium film if there are temperature gradients in the film which produce a thermodynamic potential $\varphi_T = \int SdT$ that cancels the $\rho_s v_s^2/2\rho$ term in the hydrodynamic potential μ . If such temperature gradients exist, they would produce unacceptably large evaporation rates unless the saturated vapor pressure p_{vap} is a function of μ rather than T . If $p_{\text{vap}} = p_{\text{vap}}(\mu)$, then there should be no thinning of the film with changes in Q , and the theory is in complete agreement with Keller's results.

Kontorovich^{1,2} predicted that the thickness δ of a helium film should be given by the relationship

$$\delta = \delta_0(1-q), \quad (1)$$

where

$$\delta_0 = (\beta/gh)^{1/3}, \quad (2)$$

$$q = (\rho/6\rho_s)(Q^2/\beta)\delta_0. \quad (3)$$

The quantity δ_0 is the thickness of a stationary film calculated from a balance of gravitational and van der Waals' force potentials (h is the height above the bath and β is the van der Waals' coefficient whose numerical value for various substrates has been calculated by Schiff³). A change in the thickness $(1-q)$ due to a flow rate

$Q = \rho_s v_s \delta/\rho$ is assumed to arise because the additional term $\rho_s v_s^2/2\rho$ in the hydrodynamic potential should be balanced by a reduction in the van der Waals' potential $-\beta/\delta^3$.

Earlier experiments of Gribbon and Jackson² demonstrated that Eq. (1) predicts too large a thickness change. Recent experiments by Keller⁴ demonstrate that, to within distances of less than the thickness of an atomic layer, there is no change at all in the film thickness when the flow stops. Changes, predicted by Eq. (1) to be as large as 120 Å or roughly 30 atomic layers, did not occur.

Keller's results do, however, agree with the theoretical predictions of the thickness of a stat-

ic film. To check this agreement, note that Keller's measurements of absolute thickness (rather than thickness changes) could be off by as much as 30 Å, but the error would remain constant for all runs in a given day. (This was verified by the reproducibility of the data for any given day.) Thus one can add a constant χ (which can vary only for runs on different days) to δ so that $\delta(h) + \chi$ is exactly proportional to $h^{-1/3}$. The values of χ required for Keller's two curves (his Fig. 3) are -26 and +16 Å, well within the possible 30-Å calibration errors. Thus to within expected errors the $h^{-1/3}$ dependence is verified.

Next, use the corrected $\delta(h)$ to calculate the van der Waals' coefficient; the results are $10^6\beta/g = 4.62$ and 4.77 , very close to Schiff's³ calculated value 4.7 for silver. We have not found a calculated value for stainless steel, the material used by Keller, but $10^6\beta/g$ is calculated to be 4.3 for copper and it seems reasonable to assume that most metals should have roughly similar values. Thus we may have a good test of the theory of static films by obtaining accurate agreement between the experimental and theoretical values of the van der Waals' coefficients. (It would be useful to have new calculations of β for stainless steel, and for a silver film over glass which may be used in later experiments.)

To explain the dynamic behavior of a film, we will assume the validity of the standard superfluid equation written in the form

$$\dot{\vec{v}}_s = -\nabla\left(\frac{p}{\rho} + \frac{\rho_s v_s^2}{2\rho} + gy - \frac{\beta}{z^3} - \varphi_T\right) + \vec{v}_s \times (\nabla \times \vec{v}_s), \quad (4)$$

where \vec{v}_s is the superfluid velocity field, $\dot{\vec{v}}_s = \partial\vec{v}_s/\partial t$, y is distance in the vertical direction, z is the perpendicular distance out from the beaker wall, and φ_T is the temperature-dependent potential defined by⁵

$$\nabla\varphi_T = S\nabla T; \quad \varphi_T = \int_{T_1}^{T_2} S(T)dT. \quad (5)$$

Various effects, which should have no bearing on the relationship between the flow rate Q and the film structure, have been neglected. We neglected possible variations of entropy with film thickness δ because of Keller's result that δ is independent of Q . Likewise we have neglected surface-tension forces caused by curvature of the surface because the curvature should also be independent of Q . Finally, we have neglected evaporation and condensation effects because current theories assume that evaporation rates are

independent of uniform motion of the liquid surface (we will return to this point later).

To apply Eq. (4) to a flowing film, we will first set $p/\rho = p_0/\rho$ at the surface. This neglects changes in gas pressure due to gravitational and van der Waals' forces, changes that can significantly affect the chemical potential of the gas ($\Delta p/\rho_{\text{gas}}$) but not the liquid ($\Delta p/\rho_{\text{liquid}}$) because of the much greater density of the liquid.

Second, we will use the fact that at all times the liquid level in the potential probe,⁶ located just below the thickness-measuring capacitor, matched the liquid level inside the bucket, indicating that all the potential drop occurred beyond the probe. According to a current interpretation of energy dissipation and vortex motion in superfluids,⁷ the absence of a potential drop indicates that the $\vec{v}_s \times (\nabla \times \vec{v}_s)$ term should not contribute to Eq. (4). This will happen either because there is no vorticity in the film between the bucket and the probe, or because the vortex lines are moving with the fluid particles, giving rise to a $\dot{\vec{v}}_s$ term that just cancels the $\vec{v}_s \times (\nabla \times \vec{v}_s)$ term. For simplicity we will assume that the probe indicates there is no vorticity between the bucket and the thickness-measuring capacitor. (As pointed out by Keller, in some but not all runs there might be a potential drop and vorticity inside the capacitor, just above the probe. If this occurred it had no effect on the experimental measurement of d , and thus will be ignored.)

Finally, noting that the observed flow rates Q were constant until the flow stopped, we set $\dot{\vec{v}}_s = 0$ in Eq. (4) with the result

$$\nabla\mu \equiv \nabla(p/\rho + \rho_s v_s^2/2\rho + gy - \beta/z^3 - \varphi_T) = 0, \quad (6)$$

where μ , which we will call the "hydrodynamic potential" is defined by the superfluid hydrodynamic equation.

The general solution of Eq. (6) is $\mu(y, z) = f(t)$, where $f(t)$ is a spatially constant function of time. Comparing $\mu(y, z)$ at point 1 on the surface of the liquid in the bucket ($p_1 = p_0$, $v_{s1} = 0$, $y_1 = 0$, $\varphi_{T1} = 0$, and $z_1 \gg$ any δ in the film) with point 2 on the surface of the film inside the thickness-measuring capacitor ($p_2 = p_0$, $v_{s2} = v_s$, $y_2 = h$, $z_2 = \delta$), we get

$$(\rho_s v_s^2/2\rho) + gh - (\beta/\delta^3) - \int_{T_1}^{T_2} S(T)dT = 0. \quad (7)$$

Equation (7) is by no means a new result; the point of our discussion is that a failure of Eq. (7) would require a re-evaluation of our assumptions about the hydrodynamics of superfluid in a helium film.

In checking Eq. (7), we note first that it is in

agreement with Keller's results for a static film ($v_s = 0$) if we set $T_2 = T_1$ —i.e., if there are no temperature gradients in the film. Kontorovich assumed that there would be no temperature gradients in a flowing film, and was thus led to the equation

$$(\rho_s v_s^2 / 2\rho) + gh - \beta / \delta^3 = 0, \quad (8)$$

whose solution is given in Eqs. (1), (2), and (3). As we have mentioned, Eq. (8) does not agree with experiment and thus must be rejected.

Rather than introducing additional assumptions about temperature gradients or about the nature of the equilibrium between the liquid and gas at the free surface, we can instead combine Keller's experimental results that $gh = \beta / \delta^3$ with Eq. (7) to obtain

$$\int_{T_1}^{T_2} S(T) dT = \rho_s v_s^2 / 2\rho = \rho Q^2 / 2\rho_s \delta^2. \quad (9)$$

Equation (9) predicts a flow-dependent temperature gradient in the helium film, with temperature variations ΔT (between the bath and a point of thickness δ on the film) shown in Table I. We have calculated ΔT (in mdeg) for $Q/\delta = 90$ cm/sec (corresponding to a point about 7 cm above the bath in Keller's experiment) and $Q/\delta = 28.4$ cm/sec (which should correspond to a similar height h for the flow of helium over glass; the exact value of 28.4 was chosen because Q^2/δ^2 is 10 times smaller than for 90 cm/sec).

It is important to note that Keller's experiments were carried out for temperatures between 1.1°K and T_λ . As of now there is no evidence that the thickness is independent of Q below temperatures of 1.1°K; thus, a failure to observe the large temperature rises when the bath is well below 1°K would not necessarily be in contradiction with the superfluid hydrodynamic equation we have used. [The slight rise in ΔT near λ point in Table I is misleading, for by calculating with constant Q/δ we have assumed an inordinately high $v_s = (\rho/\rho_s)(Q/\delta)$.]

In the temperature range 1.1 to 2°K, the predicted temperature rises ranging from 0.1 to 10 mdeg should be rather easily measured. However, one should be careful not to use a thermometer that alters the local flow rate. For example, a thermometer consisting of a somewhat rough painted strip could cause a several-fold increase in the helium film thickness and a drop in $(Q/\delta)^2$ by as much as a factor of 10. An evaporated superconducting bolometer is recommended for experimental investigation of ΔT . In addition

Table I. Predicted temperature variations in a flowing helium film.

T_{bath} (°K)	ΔT ($Q/\delta = 90$) (mdeg)	ΔT ($Q/\delta = 28.4$) (mdeg)
0	670	378
0.2	472	185
0.4	290	62
0.6	158	23
0.8	61	8.5
1.0	20.8	2.41
1.1	12.3	1.33
1.2	7.75	0.789
1.4	3.26	0.328
1.6	1.67	0.167
1.8	1.08	0.108
2.0	0.952	0.0952
2.15	2.15	0.215

tion the bolometer should be located so that there is no potential drop (as measured by a Hammel-Keller potential probe) between the bolometer and the bath.

If the flow-dependent temperature gradients listed in Table I are observed, we will have to revise our theory of evaporation from a thin film. Using the theoretical and experimental results of Atkins, Rosenbaum, and Seki⁸ for the evaporation rate resulting from a reduction of the gas pressure below the saturated vapor pressure p_{vap} by an amount $\Delta p = p_{\text{vap}} - p$, and calculating Δp by

$$\Delta p = [\partial p_{\text{vap}}(T) / \partial T] \Delta T, \quad (10)$$

where $\Delta T = 7.75$ mdeg K is the predicted temperature rise at $T = 1.2^\circ\text{K}$ for $Q/\delta = 90$ cm/sec, we find that the evaporation rate from a hot section of the film is an order of magnitude greater than the flow rates in the film. Such evaporation rates simply cannot exist. In fact if we had begun our discussion with a study of evaporation rates we would have concluded that temperature variations greater than a few microdegrees could not exist in the film, that the φ_T term in μ must be negligible, and that a moving film must become thinner as predicted by Kontorovich.

To explain Keller's results that the film does not become thinner, we propose that Eq. (10) is in error, and that the vapor pressure p_{vap} above the film is a function of the potential μ rather than just the temperature T . If, even in a moving film, μ is the energy required to remove a unit mass of fluid from the film, then it is not unreasonable to suggest that evaporation processes are related to μ rather than T .

If $p_{\text{vap}} = p_{\text{vap}}(\mu)$, then our hydrodynamic requirement that μ be constant along the surface of the film [Eq. (7)] means that $p_{\text{vap}}(\mu)$ is constant at the surface. Thus the actual pressure p must also be constant (and equal to p_{vap}) if there is to be no evaporation or condensation.

It is likely that in a steady state variations in gas pressure are caused only by gravitational and van der Waals' forces, and that pressure variations should be given by the equation

$$p = p_{\text{vap}} e^{-m\varphi/kT}, \quad \varphi = gh - \beta/z^3, \quad (11)$$

where p_{vap} is the vapor pressure at a point just above the bath, h and z are distances above the bath and out from the wall, respectively. Our new condition that $p = p_{\text{vap}}(\mu) = \text{const}$, all along the surface, requires $\varphi = 0$ in Eq. (11), or we must have

$$gh = \beta/\delta^3, \quad (12)$$

which is precisely Keller's experimental result.

In summary, we point out that an observation of the temperature variations predicted in Table I can be considered (1) a powerful experimental check of the applicability of the superfluid hydrodynamic equation [Eq. (4)] to helium films; (2) a fairly reliable check of our assumption that

$$p_{\text{vap}} = p_{\text{vap}}(\mu), \quad (13)$$

because we would otherwise have a very difficult time explaining why the film did not evaporate; and (3) such temperature variations, aside from giving a direct experimental measurement of the entropy of a thin film, could be used for rather detailed investigations of film flow velocities.

If the predicted temperature variations are not observed there could be one, somewhat unlikely, explanation consistent with the theory discussed here. Perhaps the superconducting bolometer as well as the vapor pressure is sensitive to chang-

es in potential μ rather than temperature T . If this is so, one should observe the same changes in potential with a bolometer as with a Hammel-Keller potentiometer probe. A series of bolometers and probes placed side by side spanning the region where $\Delta\mu$ is observed should easily check this prediction. If it turns out that the bolometer shows no appreciable variation at all along the film, then a new theory of superfluid hydrodynamics in a helium film will be needed.

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¹V. M. Kontorovich, Zh. Eksperim. i Teor. Fiz. **30**, 805 (1956) [Soviet Phys. JETP **3**, 770 (1956)]. [Note: replace V_s by \vec{v}_s in Eq. (1)].

²P. W. F. Gribbon and L. C. Jackson, Can. J. Phys. **41**, 1047 (1963).

³See, for example, K. R. Atkins, in Progress in Low-Temperature Physics, edited by C. J. Gorter (North-Holland, Amsterdam, 1947), Vol. II, p. 131.

⁴W. E. Keller, preceding Letter [Phys. Rev. Letters **24**, 569 (1970)].

⁵That entropy may also be a function of the film thickness δ , i.e., $S = S(\delta, T)$, is a point emphasized by Goodstein in a study of third sound in unsaturated films. [D. Goodstein, Phys. Rev. **183**, 327 (1969).]

⁶For a description of the potential probes, see W. E. Keller and E. F. Hammel, Phys. Rev. Letters **17**, 998 (1966).

⁷See E. Huggins, Phys. Rev. (to be published). The theorems discussed neglect temperature effects and possible surface effects that could occur in a film. Thus such effects might allow the $\vec{v}_s \times (\nabla \times \vec{v}_s)$ term to contribute to Eq. (4) without a potential drop being observed.

⁸K. R. Atkins, B. Rosenbaum, and H. Seki, Phys. Rev. **113**, 751 (1959).