ELECTROMAGNETIC INTERACTIONS WITH AN ARBITRARY LOOSELY BOUND SYSTEM*

Richard A. Krajcik and Leslie L. Foldy
Department of Physics, Case Western Reserve University, Cleveland, Ohio 44106
(Received 8 January 1970)

We present an electromagnetic interaction Hamiltonian for an arbitrary N-particle system with any mass, charge, spin, and magnetic moment which is consistent with the low-energy theorem for Compton scattering and the Drell-Hearn-Gerasimov sum rule to order $(\text{mass})^{-2}$. It is related to the sum of individual Foldy-Wouthuysen interactions by a unitary transformation.

The correct form of the electromagnetic interaction for a system of loosely bound nonrelativistic particles has not ever been seriously in question, at least in those cases where "exchange" currents play no role. If, however, one is interested in terms of relativistic order, i.e. $(\text{mass})^{-2}$, then it has previously been assumed that the correct interaction for a collection of particles is obtained from the usual Foldy-Wouthuysen (FW) reduction for one particle in an electromagnetic field by summing the one-particle interactions over all the particles of the system. This procedure recently came into serious question when it was pointed out by Barton and Dombey[1] that for loosely bound systems it yields results for absorption cross sections which violate the Drell and Hearn[2] and Gerasimov[3] (DHG) sum rule. It was further remarked by Barton[4] that the problem arose from the failure of this form of the interaction to yield the correct low-energy theorem (LET) for Compton scattering first established by Low[5] and Gell-Mann and Goldberger[6] and later extended by others.[7,8]

The seriousness of this shortcoming is emphasized by the observation that the LET and DHG sum rule depend only on fundamental assumptions of Lorentz covariance, gauge invariance, and time-reversal invariance. The LET also assumes an energy gap separating the state in which elastic Compton scattering occurs [for brevity we call this the "ground" state of the system, which usually it is, but need not be] and any other state except those belonging to the same angular-momentum multiplet, while the DHG sum rule also requires that there be no subtraction constant in the dispersion relation for the spin-dependent part of the forward Compton scattering amplitude.

Theorems of such impeccable lineage clearly demand proper respect by electromagnetic (EM) interaction Hamiltonians. Recently, Osborn[9] and Brodsky and Primack[10] have proposed EM-interaction Hamiltonians purportedly free of these difficulties. In addition to the usual FW terms there are additional two-particle terms which (interestingly!) are strongly nonlocal with respect to the nonrelativistic coordinates of the particles. In the present Letter we extend and clarify these important results by submitting the results of calculations which show: (a) The Hamiltonian of Brodsky and Primack yields results consistent with the LET and DHG sum rule only when the magnetic moment of the system arises purely from spin contributions. (Note added after submittal: This fact was recognized by Brodsky and Primack in Ref. 2 of their second paper. It was overlooked by the present authors since they were working from a preliminary version of the preprint of that paper.) (b) Terms which Osborn obtains but removes as not contributing to the spin-dependent amplitude in the special case he considers can in fact contribute to the spin-dependent amplitude and are necessary to obtain consistency with the LET and DHG sum rule when part of the magnetic moment of the system arises from orbital motion. (c) The results can be extended to systems composed of any number of particles with any spins, charges, and magnetic moments (intrinsic or normal)[11] to obtain an EM-interaction Hamiltonian to order $(\text{mass})^{-2}$ which is consistent with the theorem and sum rule. (d) The Hamiltonian is related to the FW reduced Hamiltonian by a unitary transformation. Its uniqueness and extensibility to higher order in inverse masses are still in question and for their clarification require the analysis of problems associated with interparticle interactions, exchange currents, and possibly the form of higher-moment interactions.

Interaction Hamiltonian.—The usual FW reduction of the Hamiltonian describing the interaction of a collection of N loosely bound particles of spin 0, $\frac{1}{2}$, or 1 with an electromagnetic field yields the following result[12-14]:

$$H_{\mathrm{FW}}^{\mathrm{EM}}=H_{\mathrm{FW}}^{(1)}+H_{\mathrm{FW}}^{(2)},$$

with

$$H_{\mathrm{FW}}^{(1)}=\sum\left\{\epsilon_\mu\varphi(\vec{\mathrm{r}}_\mu)-\frac{\epsilon_\mu}{m_\mu}\vec{\mathrm{p}}_\mu\circ\vec{\mathrm{A}}(\vec{\mathrm{r}}_\mu)+\frac{\epsilon_\mu^{\,2}}{2m_\mu}A^2(\vec{\mathrm{r}}_\mu)-g_\mu^{\,s}\vec{\mathrm{s}}_\mu\cdot\vec{\mathrm{H}}(\vec{\mathrm{r}}_\mu)\right.$$
$$\left.-(g_\mu^{\,s}-g_\mu^{\,l})\vec{\mathrm{s}}_\mu\cdot\left[\vec{\mathrm{E}}(\vec{\mathrm{r}}_\mu)\times\frac{\vec{\mathrm{p}}_\mu-\epsilon_\mu\vec{\mathrm{A}}(\vec{\mathrm{r}}_\mu)}{2m_\mu}-\frac{\vec{\mathrm{p}}_\mu-\epsilon_\mu\vec{\mathrm{A}}(\vec{\mathrm{r}}_\mu)}{2m_\mu}\times\vec{\mathrm{E}}(\vec{\mathrm{r}}_\mu)\right]\right\},$$

where

$$g_\nu^{\,l}\equiv\epsilon_\nu/2m_\nu,\quad g_\nu^{\,s}\equiv\left[\frac{\mu_\nu^{\,0}}{s_\nu}+\frac{\epsilon_\nu}{2m_\nu s_\nu}\right],$$

except $g_\nu^{\,s}\equiv0$ for $s_\nu=0$. Here m_ν, ϵ_ν, $\mu_\nu^{\,0}$, and s_ν represent the mass, charge, intrinsic magnetic moment, and intrinsic (canonical) spin of particle ν, respectively, and the radiation gauge $\vec{\nabla}_\nu\circ\vec{\mathrm{A}}(\vec{\mathrm{r}}_\nu)=0$ is assumed. All summations will be over the index μ and will represent sums over all the particles.

$H_{\mathrm{FW}}^{(2)}$ contains the collection of Darwin terms and quadrupole-moment terms. Higher-spin particles are expected to retain the form presented in $H_{\mathrm{FW}}^{(1)}$, but with additional higher-moment terms present in $H_{\mathrm{FW}}^{(2)}$.

In the absence of interaction, the canonical relativistic dynamical variables[15] $\vec{\mathrm{r}}_\nu$, $\vec{\mathrm{p}}_\nu$, and $\vec{\mathrm{s}}_\nu$ may be separated into external or (c.m.) variables $\vec{\mathrm{R}}$ and $\vec{\mathrm{P}}$, and internal variables $\vec{\rho}_\nu$, $\vec{\pi}_\nu$, and $\vec{\sigma}_\nu$.[16,17] The variables $\vec{\rho}_\nu$ and $\vec{\pi}_\nu$ are not all independent and satisfy the following conditions: $\sum\vec{\pi}_\mu=0$, and to order $(\text{mass})^{-2}$, $\sum m_\mu\vec{\rho}_\mu/M_T=0$, where $M_T\equiv\sum m_\mu$. The internal spin variable $\vec{\sigma}_\nu$ is defined to be the intrinsic (canonical) spin $\vec{s}_\nu$ as observed in the c.m. frame of the system rather than the c.m. frame of particle ν. The total spin of the system is then defined to be

$$\vec{\mathrm{S}}\equiv\sum(\vec{\rho}_\mu\times\vec{\pi}_\mu+\vec{\sigma}_\mu).$$

At least to order $(\text{mass})^{-2}$, the internal variables $\vec{\rho}_\nu$, $\vec{\pi}_\nu$, and $\vec{\sigma}_\nu$ commute with the external variables $\vec{\mathrm{R}}$ and $\vec{\mathrm{P}}$, and except for the fact that the variables $\vec{\rho}_\nu$ and $\vec{\pi}_\nu$ are not all independent, behave like $\vec{\mathrm{r}}_\nu$, $\vec{\mathrm{p}}_\nu$, and $\vec{\mathrm{s}}_\nu$, as indicated by the commutation relations below:

(i) $[\rho_i^{\,\nu},\sigma_j^{\,\mu}]=[\pi_i^{\,\nu},\sigma_j^{\,\mu}]=[\rho_i^{\,\nu},\rho_j^{\,\mu}]=[\pi_i^{\,\nu},\pi_j^{\,\mu}]=0$,

(ii) $[\pi_i^{\,\nu},\rho_j^{\,\mu}]=-i[\delta_{\nu\mu}-m_\nu/M_T]\delta_{ij}$,

(iii) $[\sigma_i^{\,\nu},\sigma_j^{\,\mu}]=i\delta_{\nu\mu}\epsilon_{ijk}\sigma_k^{\,\nu}$,

(iv) $[S_i,S_j]=i\epsilon_{ijk}S_k$,

(v) $[S_i,\nu_j^{\,\mu}]=i\epsilon_{ijk}\nu_k^{\,\mu}$ for $\vec{\nu}_\mu=\vec{\rho}_\mu,\vec{\pi}_\mu,\vec{\sigma}_\mu$.

$\vec{\mathrm{R}}$ and $\vec{\mathrm{P}}$ satisfy the commutator $[P_i,R_j]=-i\delta_{ij}$, where $\vec{\mathrm{P}}\equiv\sum\vec{\mathrm{p}}_\mu$.

Together $\vec{\rho}_\nu$, $\vec{\pi}_\nu$, $\vec{\sigma}_\nu$, $\vec{\mathrm{R}}$, and $\vec{\mathrm{P}}$ form a complete set in the same sense as $\vec{\mathrm{r}}_\nu$, $\vec{\mathrm{p}}_\nu$, and $\vec{\mathrm{s}}_\nu$; i.e., the only operators which commute with all the former are operators which commute with all the latter, and conversely.

In terms of $\vec{\rho}_\nu$, $\vec{\pi}_\nu$, $\vec{\sigma}_\nu$, $\vec{\mathrm{R}}$, and $\vec{\mathrm{P}}$, a Hermitean operator χ may be defined:

$$\chi\equiv-\frac{1}{2}\sum\frac{\vec{\rho}_\mu\cdot\vec{\mathrm{P}}\vec{\pi}_\mu\cdot\vec{\mathrm{P}}}{2M_T^{\,2}}-\frac{1}{2}\sum\frac{\vec{\rho}_\mu\cdot\vec{\mathrm{P}}\pi_\mu^{\,2}}{2m_\mu M_T}+\text{H.c.}+\sum\frac{\vec{\sigma}_\mu\times\vec{\pi}_\mu\cdot\vec{\mathrm{P}}}{2m_\mu M_T}.$$

χ is proportional to $\Phi^{(1)\prime}$ of Foldy[18] (H.c. denotes Hermitean conjugate).

To order $(\text{mass})^{-2}$, the unitary transformation $e^{i\chi}$ relates the lowest-order (Galilean) position, momentum, and internal spin variables of particle ν to the canonical variables $\vec{\mathrm{r}}_\nu$, $\vec{\mathrm{p}}_\nu$, and $\vec{\mathrm{s}}_\nu$[18]:

$$\vec{\mathrm{r}}_\nu=e^{i\chi}(\vec{\rho}_\nu+\vec{\mathrm{R}})e^{-i\chi},\quad\vec{\mathrm{p}}_\nu=e^{i\chi}[\vec{\pi}_\nu+(m_\nu/M_T)\vec{\mathrm{P}}]e^{-i\chi},\quad\vec{\mathrm{s}}_\nu=e^{i\chi}\vec{\sigma}_\nu e^{-i\chi}.$$

To order $(\text{mass})^{-2}$, these are the variables given by Bakamjian and Thomas.[16] Together with the subsidiary conditions, these expressions may serve to define $\vec{\rho}_\nu$, $\vec{\pi}_\nu$, $\vec{\sigma}_\nu$, and $\vec{\mathrm{R}}$ to order $(\text{mass})^{-2}$.

In the presence of electromagnetic fields, $\vec{p}_\nu$ is replaced by the gauge-invariant momentum $\tilde{p}_\nu \equiv \vec{p}_\nu - \epsilon_\nu \vec{A}(\vec{r}_\nu)$. Hence, $\tilde{P} \equiv \sum \tilde{p}_\mu = \vec{P} - \sum \epsilon_\mu \vec{A}(\vec{r}_\mu)$, and $\tilde{\pi}_\nu \equiv \tilde{p}_\nu - m_\nu \tilde{P}/M_T = \vec{\pi}_\nu - \epsilon_\nu \vec{A}(\vec{r}_\nu) + m_\nu \sum \epsilon_\mu \vec{A}(\vec{r}_\mu)/M_T$. χ becomes $\tilde{\chi} \equiv \chi(\tilde{\pi}_\nu, \tilde{P})$, and the unitary transformation $e^{i\tilde{\chi}}$ of $H_{\rm FW}{}^{\rm EM}$ yields the desired EM-interaction Hamiltonian

$$H_{\rm IR}{}^{\rm EM} = e^{i\tilde{\chi}} H_{\rm FW}{}^{\rm EM} e^{-i\tilde{\chi}} - ie^{i\tilde{\chi}}(\partial e^{-i\tilde{\chi}}/\partial t),$$

where IR denotes infrarelativistic. To order $(\text{mass})^{-2}$, $H_{\rm IR}{}^{\rm EM}$ has the form

$$H_{\rm IR}{}^{\rm EM} = H_{\rm FW}{}^{\rm EM} + H_{\Delta\rm FW}{}^{(1)}, \quad H_{\Delta\rm FW}{}^{(1)} = -\tfrac{1}{2}\sum \epsilon_\mu [\vec{E}(\vec{r}_\mu)\cdot\tilde{a}_\mu + \tilde{a}_\mu \cdot \vec{E}(\vec{r}_\mu)],$$

$$\tilde{a}_\nu \equiv -\tfrac{1}{2}\tilde{\rho}_\nu \cdot \frac{\tilde{P}}{M_T}\left[\frac{\tilde{\pi}_\nu}{m_\nu} + \frac{\tilde{P}}{2M_T}\right] - \frac{1}{2}\sum \frac{\tilde{\pi}_\mu{}^2 \tilde{\rho}_\mu}{2m_\mu M_T} + \frac{1}{2}\sum \frac{(\tilde{\rho}_\mu \times \tilde{\pi}_\mu)\times \tilde{P}}{2M_T{}^2} + \text{h.c.} - \frac{\vec{\sigma}_\nu \times \tilde{P}}{2m_\nu M_T} + \sum \frac{\vec{\sigma}_\mu \times \tilde{\pi}_\mu}{2m_\mu M_T} + \sum \frac{\vec{\sigma}_\mu \times \tilde{P}}{2M_T{}^2}.$$

It has been assumed that $\mu_\nu{}^0 \propto 1/m_\nu$.

It should be noted that $H_{\rm IR}{}^{\rm EM}$ could equivalently be obtained by the procedure of Osborn[9]; namely, the replacement of $\vec{r}_\nu$ by $\vec{r}_\nu{}^{\rm NR} + \tilde{a}_\nu$ in $H_{\rm FW}{}^{\rm EM}$ (NR denotes nonrelativistic), which is then expressed in terms of $\vec{r}_\nu{}^{\rm NR}$ with $-\vec{\nabla}_\nu \varphi(\vec{r}_\nu)$ being replaced by $\vec{E}(\vec{r}_\nu)$. $\tilde{a}_\nu$ is given above, and $\vec{r}_\nu{}^{\rm NR} \equiv \vec{\rho}_\nu + \vec{R}$.

$H_{\Delta\rm FW}{}^{(1)}$ represents the "correction" terms to the FW reduction. We remark that for two particles the "correction" terms do not reduce simply to the extra terms retained previously by Osborn[9] and by Brodsky and Primack,[10] but include their terms plus terms needed to describe the role of internal orbital motion of the system. Even in the case where both particles are spin zero, the "correction" terms are not zero, and they are necessary in order to maintain consistency with the LET and the DHG sum rule.

The low energy theorem and DHG sum rule.— The scattering amplitude for Compton scattering can be determined from general principles and, to first order in photon frequency, it depends only on the mass, charge, magnetic moment, and spin of the target.[5-8] Our calculations are patterned after the procedure used by Barton[4] and Osborn,[9] and only special points of interest will be noted here.

It is explicitly assumed that an energy gap exists between the ground state and any excited state, that the only degeneracy of the ground state is the usual multiplet structure associated with rotational invariance, and that the initial momentum of the composite system is zero.

To first order in photon frequency, $H_{\rm FW}{}^{(2)}$ does not contribute to the scattering amplitude. Those terms which do contribute either are or can be reduced by the use of closure to matrix elements of operators between states belonging to the ground-state multiplet. Since the internal variables are irreducible vector operators under rotations generated by $\vec{S}$, the Wigner-Eckart theorem may be applied together with time-reversal invariance to remove many of these terms until finally only matrix elements of $\vec{l}_\mu \equiv \vec{\rho}_\mu \times \vec{\pi}_\mu$ and $\vec{\sigma}_\mu$ require evaluation. The relationship of these to the magnetic moment of the system is secured by defining the following reduced matrix-element ratios:

$$\Omega_\nu{}^l(\alpha, S) = \frac{\langle \alpha S \| l_\nu \| \alpha S\rangle}{\langle \alpha S \| S \| \alpha S\rangle}, \quad \Omega_\nu{}^\sigma(\alpha, S) = \frac{\langle \alpha S \| \sigma_\nu \| \alpha S\rangle}{\langle \alpha S \| S \| \alpha S\rangle},$$

where S represents the spin of the system and α represents all other quantum numbers. All the structure of the composite system relevant to our calculation, i.e., the orientation and coupling of spins and orbital angular momentum, is now contained in $\Omega_\nu{}^l(\boldsymbol{\alpha}, S)$ and $\Omega_\nu{}^\sigma(\boldsymbol{\alpha}, S)$, and therein lies their power. Through the Wigner-Eckart theorem the magnetic moment of the composite system (assuming no "exchange" currents) can be expressed in terms of $\Omega_\nu{}^l(\alpha, S)$ and $\Omega_\nu{}^\sigma(\alpha, S)$: With $\vec{\mu} = \sum[g_\mu{}^l \vec{l}_\mu + g_\mu{}^s \vec{\sigma}_\mu]$ and $g_\nu{}^l$ and $g_\nu{}^s$ given earlier,

$$\mu \equiv gS = \langle \alpha S, M=S|\mu_z|\alpha S, M=S\rangle = \sum[g_\mu{}^l \Omega_\mu{}^l + g_\mu{}^s \Omega_\mu{}^\sigma]S.$$

It is in this manner that the magnetic moment of the composite system finally enters the Compton amplitude.

We emphasize that time-reversal invariance must be explicitly adjoined to gauge invariance and Lorentz invariance to establish the theorem, as is to be expected; space-inversion invariance (as used by Osborn and by Brodsky and Primack) is not required to derive the LET.[19]

The verification of the DHG sum rule follows the approach and techniques of Barton and Dombey[1] and

Osborn[9] as well as those indicated above for the LET. Once again $H_{\Delta FW}^{(1)}$ must be used in its entirety to take proper account of orbital-motion contributions, and time-reversal invariance, but not space-inversion invariance, must be explicitly assumed to obtain consistency with the DHG sum rule.

It should be remarked that had Osborn[9] been uniform in applying his substitution principle, he would have obtained the interaction corresponding to the two-particle form of our result, and had Brodsky and Primack[10] retained first-order retardation effects which contribute to the description of orbital motion, they too probably would have obtained the same two-particle interaction Hamiltonian.

Clearly, similar considerations must obtain, mutatis mutandis, for other interactions of loosely bound systems and, in particular, for the weak-interaction Hamiltonian.

The authors would like to thank Dr. Paul Kantor for useful discussions, and one of the authors (R.A.K.) would like to express his gratitude for an National Defense Education Act fellowship grant.

---

*Work supported in part by the U. S. Atomic Energy Commission.

[1]G. Barton and N. Dombey, Phys. Rev. 162, 1520 (1967).

[2]S. D. Drell and A. C. Hearn, Phys. Rev. Letters 16, 908 (1966).

[3]S. B. Gerasimov, Yadern. Fiz. 2, 598 (1965) [Soviet J. Nucl. Phys. 2, 430 (1966)].

[4]G. Barton, to be published.

[5]F. E. Low, Phys. Rev. 96, 1428 (1954).

[6]M. Gell-Mann and M. L. Goldberger, Phys. Rev. 96, 1433 (1954).

[7]L. I. Lapidus and Chou K.-C., Zh. Eksperim. i Teor. Fiz. 39, 1286 (1960) [Soviet Phys. JETP 12, 898 (1961)].

[8]I. J. Kalet, Phys. Rev. 176, 2135 (1968).

[9]H. Osborn, Phys. Rev. 176, 1523 (1968).

[10]S. J. Brodsky and J. R. Primack, Phys. Rev. 174, 2071 (1968), and Ann. Phys. 52, 315 (1969).

[11]L. L. Foldy, Phys. Rev. 87, 688 (1952).

[12]L. L. Foldy and S. A. Wouthuysen, Phys. Rev. 78, 29 (1950).

[13]K. M. Case, Phys. Rev. 95, 1323 (1954).

[14]J. A. Young and S. A. Bludman, Phys. Rev. 131, 2326 (1963).

[15]L. L. Foldy, Phys. Rev. 102, 568 (1956).

[16]B. Bakamjian and L. H. Thomas, Phys. Rev. 92, 1300 (1953).

[17]H. Osborn, Phys. Rev. 176, 1514 (1968).

[18]L. L. Foldy, Phys. Rev. 122, 275 (1961). The relevant equation (Eq. 66) contains the following misprint: S_ν should be replaced with $2s_\nu$.

[19]L. D. Landau, Zh. Eksperim. i Teor. Fiz. 32, 405 (1957) [Soviet Phys. JETP 5, 336 (1957)], and Nucl. Phys. 3, 127 (1957).