an independent determination of the coupling constant.

From the assembled astrophysical data it is concluded, first, that the $(\bar{e}\nu_e)(\bar{\nu}_e e)$ interaction does exist in nature, and, second, that the value of the coupling constant is equal to, or close to, the coupling constant of beta decay, namely, $g^2 = 10^{0\pm 2} g_\beta^2$.

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[1]R. P. Feynman and M. Gell-Mann, Phys. Rev. 109, 193 (1958).
[2]M. A. Ruderman, Rept. Progr. Phys. 28, 411 (1965); H.-Y. Chiu, Ann. Rev. Nucl. Sci. 16, 591 (1966).
[3]J. B. Adams, M. A. Ruderman, and C. H. Woo, Phys. Rev. 129, 1383 (1963).
[4]C.-W. Chin, H.-Y. Chiu, and R. Stothers, Ann. Phys. 39, 280 (1966).
[5]R. Stothers, Astron. J. 71, 943 (1966).
[6]V. Weidemann, Ann. Rev. Astron. Astrophys. 6, 351 (1968).
[7]O. J. Eggen, Astrophys. J. 157, 287 (1969).
[8]O. J. Eggen and J. L. Greenstein, Astrophys. J. 141, 83 (1965).
[9]H. M. Van Horn, Astrophys. J. 151, 227 (1968).
[10]C.-W. Chin and R. Stothers, to be published.
[11]S. C. Vila, in Low-Luminosity Stars, edited by S. S. Kumar (Gordon and Breach, New York, 1969), p. 351.
[12]S. C. Vila, Astrophys. J. 146, 437 (1966), and 149, 613 (1967), and private communication; M. P. Savedoff, H. M. Van Horn, and S. C. Vila, Astrophys J. 155, 221 (1969).
[13]A. Sandage, Astrophys. J. 125, 422 (1957).
[14]L. G. Henyey, R. LeLevier, and R. D. Levee, Astrophys. J. 129, 2 (1959); B. Strömgren, in Stellar Evolution, edited by R. F. Stein and A. G. W. Cameron (Plenum, New York, 1966), p. 391; U. Lindoff, Arkiv Astron. 5, 1 (1968); R. P. Kraft, in Stellar Astronomy, edited by H.-Y. Chiu, R. L. Warasila, and J. L. Remo (Gordon and Breach, New York, 1969), Vol. 1, p. 317.
[15]R. Stothers, Astrophys. J. 155, 935 (1969).
[16]R. Stothers and C.-W. Chin, Astrophys. J. 158, 1039 (1969).
[17]R. M. Humphreys, thesis, University of Michigan, Ann Arbor, 1969 (unpublished), and private communication.
[18]C. Hayashi, R. Hōshi, and D. Sugimoto, Progr. Theoret. Phys. (Kyoto) Suppl. No. 22, 1 (1962).
[19]R. Stothers, in Stellar Astronomy, edited by H.-Y. Chiu, R. L. Warasila, and J. L. Remo (Gordon and Breach, New York, 1969), Vol. 2, p. 205.
[20]R. Stothers, Astrophys. Letters 4, 187 (1969).
[21]R. Stothers, Astrophys. J. 138, 1085 (1963).
[22]R. Stothers and H.-Y. Chiu, Astrophys. J. 135, 963 (1962); W. A. Fowler and F. Hoyle, Astrophys. J. Suppl. 9, 201 (1964).
[23]F. Reines, private communication.

SIMPLE APPROACH TO UNITARIZATION IN HARD-MESON CALCULATIONS*

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The hard-pion effective-range formula for the pion form factor of Brehm, Golowich, and Prasad is derived from unitarity and (vector) meson dominance assuming a real tree form factor as input. The result is generalized and applied to the π-A_1-ρ system. The suitably continued finite ρ-width form factors are used to calculate the soft π^+-π^0 mass difference when (a) both tree form factors are unsubtracted in broken chiral symmetry ($\delta m \simeq 5.7$ MeV) and when (b) both tree form factors are subtracted ($\delta = -\frac{1}{2}$) ($\delta m \simeq 5.2$ MeV).

In a recent Letter[1] it was shown how the specific hard-pion current-algebra method of Ward identities[2] could be used to generate an effective-range formula for the pion form factor directly without reference to the $\pi\pi$ phase shift. Thus, the usual ordering of input and output is curiously inverted, since in this case the $T=J=1$ $\pi\pi$ phase shift δ_{11} is among the output, once the pion form factor $F(t)$ is given. Unfortunately, the derivation leading to an "on-shell dynamical equation for $F(t)$" presented in Ref. 1 largely obscures what is demonstrably an attractively simple approach to unitarization in hard-meson calculations. In our derivation it will not be necessary to tie the unitarization to any particular current-algebra procedure which produces hard-meson results. As shown below it is enough merely to require that the input hard-meson vertex, say[3]

$$\Gamma(t) = (m_\rho^2 - t)F(t), \tag{1}$$

which we shall occasionally refer to as the "tree vertex," be real. The unitarization of "tree form factors" may then be readily extended to include Veneziano-type form factors[4] with the corresponding output of low-energy phase shifts.[5]

The transformation of the familiar elastic unitarity relation,

$$\mathrm{Abs}\langle 0|V_\mu^{(3)}(0)|\pi^+(p)\pi^-(q)\text{ in}\rangle = \mathrm{Abs}\,F(t)(-p+q)_\mu = -\tfrac{1}{2}\int\langle 0|V_\mu^{(3)}(0)|\pi^+(p')\pi^-(q')\text{ in}\rangle \frac{d\vec{\mathrm{P}}d\vec{\mathrm{Q}}2W(2\pi)^4\delta(\vec{\mathrm{P}})}{[(2\pi)^6 4w(p')w(q')]}$$
$$\times\delta(W^2+(p+q)^2)\langle\pi^+(p')\pi^-(q')\text{ in}|J^{(1+i2)/\sqrt{2}}(0)|\pi^+(p)\rangle, \tag{2}$$

the starting point in our derivation, into a dynamical equation for $F(t)$ (the essential result of Ref. 1) requires the following successive approximations: (1) One keeps only the one-particle reducible part of the full amplitude[6] $\langle\pi^+(p')\pi^-(q')\text{ in}|J^{(1+i2)/\sqrt{2}}(0)|\pi^+(p)\rangle$; this is consistent with our earlier assumption that Γ is real.[7] (2) The structure of the one-particle reducible amplitude is further simplified through the assumption of meson dominance. This means vector (ρ) dominance in the present instance, but scalar (ϵ) dominance in, say, the calculation of the $T=J=0$ $\pi\pi$ phase shift.[8] Thus one finds[9]

$$\langle\pi^+(p')\pi^-(q')\text{ in}|J^{(1-i2)/\sqrt{2}}(0)|\pi^-(q)\rangle \simeq -\langle\pi^+(p')\pi^-(q')\text{ in}|V_\nu^{(3)}(0)|0\rangle g_\rho^{-2}$$
$$\times[(-\Box+m_\rho^2)\langle 0|V_\nu^{(3)}(0)|\pi^+(p)\pi^-(q)\text{ out}\rangle]_{\text{tree}}, \tag{3}$$

with

$$\mathrm{Abs}\,F(t) = |F(t)|^2\frac{Q^3}{\sqrt{t}}\,\frac{f(t)}{12\pi F_\pi^2 m_\rho^2}, \tag{4}$$

where

$$[(m_\rho^2-t)\langle 0|V_\nu^{(3)}(0)|\pi^+(p)\pi^-(q)\text{ out}\rangle]_{\text{tree}} = f(t)(-p+q)_\nu; \tag{5}$$

$$f(t) = m_\rho^2\left(1-\frac{1+\delta}{4}\frac{t}{m_\rho^2}\right) \tag{6}$$

is the linear polynomial given by hard-pion current algebra. The one-parameter solution of Eq. (4) has been treated adequately in Ref. 1[10] and is assumed in our application below. Our procedure is easily extended to several coupled two-pseudoscalar meson channels, in which case the matrix version of Eq. (4),

$$\mathrm{Abs}\,F(t) = \frac{1}{12\pi F_\pi^2 m_\rho^2}\mathrm{Tr}[F^\dagger(t)\rho(t)F(t)]f(t), \tag{7}$$

should be used. Thus one might consider treating approximately the effect of the opening of the $K\bar{K}$ channel on the high-t wing of the pion electromagnetic form factor, although we shall not do so here. Instead, we confine ourselves in the remainder of this note to some considerations relating to the effects of a finite ρ width on the matrix element $\langle 0|V_\mu^{(3)}(0)|\pi^+(p)A_1^-(q)\text{ in}\rangle$. Since the form factors $L(t)$, $M(t)$, $N(t)$ in this case, defined by

$$\langle 0|V_\mu^{(3)}(0)|\pi^+(p)A_1^-(q)\text{ in}\rangle = [L(t)\delta_{\nu\mu}+M(t)(-p+q)_\mu p_\nu+N(t)(p+q)_\mu p_\nu]i\epsilon_\nu^{(A_1)}(q), \tag{8}$$

are related to those of $\langle A_1^+(q)|V_\mu^{(3)}(0)|\pi^+(p)\rangle$ by crossing, we are enabled by analytic continuation to compute the effects of the finite ρ width on the π^+-π^0 mass difference for either unsubtracted or subtracted form factors. (The latter case has not been properly discussed in the literature for reasons of convergence.[11])

As in our earlier derivation we replace the customary ρ-dominance argument, which leads to

$$\langle 0|V_\mu^{(3)}(0)|\pi^+(p)A_1^-(q)\text{ in}\rangle$$
$$= i\int d^4x\,e^{ip\cdot x}\theta(-x_0)\sum\nolimits_\lambda\int\langle 0|V_\mu^{(3)}(0)|\rho^0(k),\lambda\rangle d\vec{\mathrm{k}}\{[2E(k)(2\pi)^3]\}^{-1}\langle\rho^0(k),\lambda|J^{(1-i2/\sqrt{2})}(x)|A_1^-(q)\rangle, \tag{9}$$

by "P-wave $\pi\pi$ dominance," with[12]

$$[\mathrm{Abs}\,L(t)-\mathrm{Abs}\,M(t)(q_0^2-p_0^2)-\mathrm{Abs}\,N(t)(q_0+p_0)^2]\epsilon_0^{(A_1)}(q) = 0, \tag{10}$$

$$\mathrm{Abs}\, L(t)\epsilon_i^{(A_1)}(q)-\mathrm{Abs}\, M(t)(q-p)_i(q_0+p_0)\epsilon_0^{(A_1)}(q)$$
$$=|F|^2\frac{Q^3}{\sqrt{t}}\frac{1}{6\pi g_\rho^2}\{(-t+m_\rho^2)L(t)\epsilon_i^{(A_1)}(q)-(-t+m_\rho^2)M(t)(q-p)_i(q_0+p_0)\epsilon_0^{(A_1)}(q)\}_{\mathrm{tree}}. \quad (11)$$

For an estimate of the effects of a finite ρ width on the π^+-π^0 mass-difference calculation for soft external pions, only $L(t)$ is needed.[13] $L(t)$ satisfies the dynamical equation

$$\mathrm{Abs}\, L(t)=|F(t)|^2\frac{Q^3(t)}{\sqrt{t}}\frac{1}{6\pi g_\rho^2}l(t), \quad (12)$$

where $l(t)=[(m_\rho^2-t)L(t)]_{\mathrm{tree}}$ is the linear polynomial (in t) given by current algebra.[14] The solution of Eq. (12) is[15]

$$L(t)=\frac{2F_\pi^2m_\rho^2}{g_\rho^2f(t)}[F(t)l(t)-F(t_0)l(t_0)]. \quad (13)$$

In the soft-external-pion limit, the π^+-π^0 mass difference is given by[16]

$$m_{\pi^+}-m_{\pi^0}=-6e^2m_\pi(2\pi)^{-4}\,\mathrm{Re}\Big(i\int\frac{d^4q}{q^2}\Big\{[F(-q^2)]^2+\frac{q^2}{m_A^2(m_A^2+q^2)}[L(-q^2)]^2\Big\}\Big), \quad (14)$$

when only contributions from the one-pion and one-A_1 meson intermediate states are kept. The soft-pion electromagnetic mass-difference calculation of Brown and Munczek[17] seems to us the aptly suited "narrow ρ-width" calculation with which to compare our finite-width approximation, i.e., Eqs. (13) and (14) with $F(-q^2)$, the continued "solution" of Eq. (4). It is easy to show that the soft-pion calculation of Ref. 17 which neglects the anomalous moment of the A_1 for convergence requires both form factors to be unsubtracted:

$$F(-q^2)=m_\rho^2/(m_\rho^2+q^2), \quad (15)$$

$$L(-q^2)=-g_\rho G_{\rho A\pi}/(m_\rho^2+q^2); \quad (16)$$

the result of "broken chiral symmetry"[17] which follows when the mass parameter m_0 of Ref. 17 is set equal to m_A is equivalently obtained by requiring that the form factor $L(-q^2)$ satisfy the current-algebra limit,[18] $L(m_A^2)=g_A/F_\pi$, so that $G_{\rho A\pi}=m_\rho^2$ and hence $g_\rho G_{\rho A\pi}=\sqrt{2}m_\rho^3$. Our finite-width analog of this model,

$$m_{\pi^+}-m_{\pi^0}=-12e^2m_\pi(2\pi)^{-4}\,\mathrm{Re}\Big\{i\int\frac{d^4q}{q^2}[F(-q^2;\delta=-1)]^2\Big(\frac{q^2+m_\rho^2}{q^2+m_A^2}\Big)\Big\}, \quad (17)$$

yields $m_{\pi^+}-m_{\pi^0}\simeq 5.7$ MeV as compared with the (narrow-width) broken-chiral-symmetry result[17] $m_{\pi^+}-m_{\pi^0}\simeq 5.1$ MeV. On the other hand, in the favored case,[1] $\delta=-\frac{1}{2}$, where both the corresponding narrow-width form factors are subtracted, we find $m_{\pi^+}-m_{\pi^0}\simeq 5.2$ MeV.

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[1]J. J. Brehm, E. Golowich, and S. C. Prasad, Phys. Rev. Letters 23, 666 (1969).

[2]H. J. Schnitzer and S. Weinberg, Phys. Rev. 164, 1828 (1967).

[3]Omitting kinematical factors, we write

$$\langle 0|V_\mu^{(3)}(0)|\pi^+(p)\pi^-(q)\ \mathrm{in}\rangle=F(t)(-p+q)_\mu,$$

with $t=W^2=-(p+q)^2$.

[4]H. Suura, Phys. Rev. Letters 23, 551 (1969); J. A. Cronin and K. Kang, Phys. Rev. Letters 23, 1004 (1969).

[5]This possibility is being actively investigated in collaboration with Professor K. Kang.

[6]M. Ida, Phys. Rev. 135, B499 (1964). It is essential that we make no distinction between stable or resonant (narrow-width) poles of the one-particle reducible amplitude; indeed, only the latter case is of interest in this note.

[7]In the case of S-wave, equal-mass (m_π) scattering in one channel, one has familiarly

$$T=K(t)(M^2-t)^{-1}\Gamma(t)+U(t),$$

with

$$\mathrm{Abs}\,\Gamma(t)=\Gamma^*(t)[(\tfrac{1}{4}t-m_\pi^2)^{1/2}/(2\sqrt{t})]U(t),\quad t\geq 4m_\pi^2,$$

and

$$U(t)=\mathfrak{N}/\mathfrak{D},$$

with K the form factor corresponding to the vertex Γ. (In the one-particle irreducible amplitude U above, the numerator function $\mathfrak{N}$ has only the unphysical cut.) Thus our neglect of the amplitude U is consistent with a real Γ; moreover our approximate T lacks a left-hand cut.

[8]In principle, the procedure could also be extended to the case where there are several particles in the same channel, e.g., ρ and ρ'.

[9]There we take $g_\rho^2/m_\rho^2=2F_\pi^2$, with $F_\pi=94$ MeV.

[10]However, note that if one tries to "simplify" the solution of the dynamical equation (4) by introducing the factored function, $G(t)=F(t)f(t)$, satisfying

$$\mathrm{Abs}\,G(t)=|G(t)|^2\frac{Q^3}{\sqrt{t}}\frac{1}{12\pi F_\pi^2 m_\rho^2},$$

then we are compelled to introduce a Castillejo-Dalitz-Dyson pole at $t=4m_\rho^2/(1+\delta)$ in G^{-1} [$G(t)$ is Herglotz while $F(t)$ is not] to remove the singularity arising from the vanishing of the linear polynomial $f(t)$. Thus, in this case,

$$F(t)=\frac{1}{f(t)}\left[\frac{1}{m_\rho^2}\frac{t}{\pi}\int_{4m_\pi^2}^{\Lambda}\left(\frac{1}{12\pi F_\pi^2 m_\rho^2}\right)\frac{Q'^3}{t'^{3/2}(t'-t)}+\beta\frac{t}{f(t)}\right]^{-1},$$

which can be reduced to the one-parameter solution for $F(t)$ given in Ref. 1.

[11]For example, Tanaka [K. Tanaka, International Centre for Theoretical Physics, Trieste, Report No. IC/68/81, 1968 (to be published)] notes that "there is no justification for dropping the quadratically divergent terms" that arise in the narrow-width calculation in this case.

[12]We use $p\cdot\epsilon^{(A_1)}(q)=(p+q)\cdot\epsilon^{(A_1)}(q)=-(p_0+q_0)\epsilon_0^{(A_1)}(q)$; note that $t=(p_0+q_0)^2$.

[13]The contributions of $M(t)$, $N(t)$ vanish in this limit. We do not consider the joint effect of mass-shell (external) pion and finite ρ width in this paper.

[14]In the hard-pion current algebra where $f(t)$ is given by Eq. (6),

$$l(t)=g_\rho g_{A\rho\pi}+(t-m_\rho^2)(g_\rho/m_\rho^2)[g_{A\rho\pi}-(m_A^2-m_\pi^2)\tfrac{1}{2}h_{A\rho\pi}],$$

with

$$g_{A\rho\pi}=\frac{m_\rho^2}{2F_\pi}\left(1+\frac{\delta}{2}-\frac{\delta}{2}\frac{m_\pi^2}{m_\rho^2}\right)\text{ and }h_{A\rho\pi}=\frac{\delta}{2F_\pi}.$$

[See, for example, S. G. Brown and G. B. West, Phys. Rev. 168, 1605 (1968).] Their constants are related to ours by $f_\rho=g_\rho$, $f_\pi=\sqrt{2}F_\pi$.

[15]The subtraction term $2F_\pi^2 m_\rho^2 F(t_0)l(t_0)/f(t)$ removes the pole at $t_0=4m_\rho^2/(1+\delta)$ introduced by the zero of $f(t)$. Note that $F(t_0)$ is real.

[16]The analytic continuation of $F(t)$ to negative t is readily obtained once the essential function

$$g(t)=\frac{2}{\pi}\frac{Q^3}{\sqrt{t}}\ln\left(\frac{\sqrt{t}+2Q}{2m_\pi}\right)-\frac{iQ^3}{\sqrt{t}},\quad t\geq 4m_\pi^2,$$

is expressed as the integral,

$$g(t)=\frac{Q^2}{\pi}\left[1-t\int_{4m_\pi^2}^{\infty}dt'\frac{Q'}{t'^{3/2}(t'-t)}\right].$$

Thus

$$g(-t)=-\frac{2}{\pi}\frac{(\tfrac{1}{4}t+m_\pi^2)^{3/2}}{\sqrt{t}}\ln\left[\frac{\sqrt{t}+2(\tfrac{1}{4}t+m_\pi^2)^{1/2}}{2m_\pi}\right],\quad t>0.$$

[17]L. M. Brown and H. Munczek, Phys. Rev. Letters 20, 680 (1968).

[18]That is,

$$\lim_{k\to 0}\langle A_1^-(p)|V_\mu^{(3)}(0)|\pi^-(k)\rangle=i(g_A/F_\pi)\epsilon_\mu^{*(A_1)}(p)=\lim_{t\to m_A^2}\{L(t)\delta_{\nu\mu}i\epsilon_\nu^{*(A_1)}(p)+\cdots\}.$$