NEW SYMMETRY FOR ACOUSTO-OPTIC SCATTERING

D. F. Nelson and M. Lax
Bell Telephone Laboratories, Incorporated, Murray Hill, New Jersey 07974
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Rotation of volume elements within an acoustic wave can make a large contribution to the scattering of light from an acoustic shear wave in optically anisotropic media, thus removing the traditionally assumed symmetry between the last two (acoustic) indices of the usual photoelastic tensor p_{ijkl}.

We wish to show by a simple argument that the traditional formulation[1] of the photoelastic effect in which the change in the inverse optical dielectric constant is taken solely proportional to the infinitesimal strain S_{kl} according to

$$(\delta k^{-1})_{ij} = p_{ij(kl)} S_{kl}, \tag{1}$$

$p_{ij(kl)}$ being the usual photoelastic tensor, is in error. The new terms needed are comparable with those already present in Eq. (1) when acoustic shear waves are used in crystals having a large optical anisotropy, such as calcite. This will necessitate a re-examination of many of the previously measured photoelastic tensor elements and a modified procedure of interpreting future acousto-optic and Brillouin scattering experiments.

Since the infinitesimal strain is defined as the symmetric combination of displacement gradients,

$$S_{kl} = \tfrac{1}{2}(u_{k,l} + u_{l,k}), \tag{2}$$

S_{kl} and hence $p_{ij(kl)}$ are symmetric upon interchange of k and l, indicated in this Letter by the parentheses around k and l on the photoelastic tensor. The antisymmetric combination of displacement gradients apparently was ignored in the traditional formulation since it corresponded simply to a rotation of the entire body when the body was in a state of uniform shear strain. The photoelastic tensor $p_{ij(kl)}$ has been assumed to govern the scattering of light from an acoustic wave by workers in the field.[2] However, in the inhomogeneous strain field of an acoustic shear wave the rotation is not a trivial rotation of the entire body but a rotation of volume elements varying within an acoustic wavelength. Moreover, in a pure shear wave the rotation is numerically equal to the strain. If the medium is optically anisotropic, a rotation of it obviously affects the propagation of light through it. Hence, an appreciable and unavoidable contribution to acousto-optic scattering should arise from the rotation effect.

Since the rotational effect is simply the change in the dielectric tensor produced by a rotation of an anisotropic crystal, we consider an infinitesimal body rotation which carries a body point at x_i of the crystal into the new position

$$x_i' = x_i + u_i = x_i + \epsilon_{ijk}\delta\theta_j x_k \equiv (\partial x_i'/\partial x_j)x_j, \tag{3}$$

where summation over repeated indices is understood, and

$$\partial x_i'/\partial x_j = \delta_{ij} + u_{i,j} = \delta_{ij} + \tilde{R}_{ij} \tag{4}$$

and

$$\tilde{R}_{ij} \equiv \tfrac{1}{2}(u_{i,j} - u_{j,i}) = -\tilde{R}_{ji} = -\epsilon_{ijk}\delta\theta_k. \tag{5}$$

$\tilde{R}_{ij}$ is the mean rotation tensor that describes the counterclockwise infinitesimal body rotation through the angle $|\delta\vec{\theta}|$ about the direction of $\delta\vec{\theta}$. The dielectric tensor κ_{ij} must transform on each index i,j under a body rotation exactly as the vector x_i transforms under the same rotation. Hence by using Eq. (4)

$$\begin{aligned}\delta\kappa_{ij} &= (\partial x_i'/\partial x_k)\,(\partial x_j'/\partial x_l)\kappa_{kl} - \kappa_{ij}, \\ &= \tilde{R}_{ik}\kappa_{kj} - \kappa_{ik}\tilde{R}_{kj}.\end{aligned} \tag{6}$$

The change in the inverse dielectric tensor, $\delta\kappa = -\kappa^{-1}\delta\kappa\,\kappa^{-1}$, is then given by

$$\begin{aligned}(\delta\kappa^{-1})_{ij} &= \tilde{R}_{ik}(\kappa^{-1})_{kj} - (\kappa^{-1})_{ik}\tilde{R}_{kj} \\ &\equiv p_{ij\,[kl]}\tilde{R}_{kl},\end{aligned} \tag{7}$$

where

$$\begin{aligned}p_{ij\,[kl]} \equiv \tfrac{1}{2}\{&(\kappa^{-1})_{il}\delta_{kj} + (\kappa^{-1})_{lj}\delta_{ik} - (\kappa^{-1})_{ik}\delta_{lj} \\ &- (\kappa^{-1})_{kj}\delta_{il}\}.\end{aligned} \tag{8}$$

The tensor $p_{ij[kl]}$ has been antisymmetrized with respect to the k and l (acoustic) indices as indicated by the brackets around k and l; it, like $p_{ij(kl)}$, is symmetric with respect to the i and j (optic) indices. The change in the inverse dielectric tensor relevant to acousto-optic scattering is the sum of Eqs. (1) and (7). Note that the tensor $p_{ij[kl]}$ can be calculated simply from the optical dielectric tensor. Note also that $p_{ij\,[kl]} \neq 0$ only when $k \neq l$; this is expected since rotation occurs only in shear waves.

It is notationally convenient to combine Eqs. (1)

and (7) and define a single new photoelastic tensor p_{ijkl}' by

$$(\delta\kappa^{-1})_{ij} = p_{ijkl}' u_{k,l}, \tag{9}$$

where

$$p_{ijkl}' = p_{ij(kl)} + p_{ij[kl]}. \tag{10}$$

It is seen here that the natural measure of elastic deformation relevant to the photoelastic effect is the displacement gradient, not the strain or the stress. The new tensor p_{ijkl}' is neither symmetric nor antisymmetric in the indices k and l. This prevents the usual contracted notation[3] for $p_{ij(kl)}$ from being applied to the k and l indices of p_{ijkl}'; it can, however, be applied to the i and j indices. We propose the following contracted notation for the k and l indices: denote (k, l) values of (1, 1), (2, 2), (3, 3), (2, 3), (3, 2), (3, 1), (1, 3), (1, 2), and (2, 1), respectively, by a single index value of 1, 2, 3, 4, $\bar{4}$, 5, $\bar{5}$, 6, and $\bar{6}$.

The number of new photoelastic tensor elements from the rotation effect depends on crystal symmetry through Eq. (8). For isotropic media and cubic crystals $p_{ijkl}' = p_{ij(kl)}$, that is, there are no new photoelastic tensor elements. For optically uniaxial crystals (hexagonal, tetragonal, and trigonal systems) $p_{ijkl}' = p_{ij(kl)}$ except that in place of p_{44} (contracted notation) there are now two tensor elements p_{44}' and $p_{4\bar{4}}'$ and in place of p_{55} there are p_{55}' and $p_{5\bar{5}}'$. For orthorhombic crystals (optically biaxial) $p_{ijkl} = p_{ij(kl)}$ except that in place of each of p_{44}, p_{55}, and p_{66} there are now two tensor elements p_{44}' and $p_{4\bar{4}}'$, p_{55}' and $p_{5\bar{5}}'$, and p_{66}' and $p_{6\bar{6}}'$, respectively. For monoclinic crystals (optically biaxial) one of the principal axes of the dielectric tensor is fixed in the x_2 crystallographic direction while the orientation of the remaining two principal axes depend upon wavelength. Thus, the dielectric tensor will possess off-diagonal elements $\kappa_{31} = \kappa_{13}$ when expressed in the crystallographic coordinate system. Also, all three diagonal elements of the dielectric tensor will be unequal. Hence, in place of each of the elements p_{44}, p_{64}, p_{15}, p_{35}, p_{55}, p_{46}, and p_{66}, there are now two tensor elements p_{44}' and $p_{4\bar{4}}'$, p_{64}' and $p_{6\bar{4}}'$, p_{15}' and $p_{1\bar{5}}'$, p_{35}' and $p_{3\bar{5}}'$, p_{55}' and $p_{5\bar{5}}'$, p_{46}' and $p_{4\bar{6}}'$, and p_{66}' and $p_{6\bar{6}}'$, respectively, giving seven additional photoelastic tensor elements for monoclinic crystals. For triclinic crystals (optically biaxial) all three of the principal coordinate axes of the dielectric tensor depend on wavelength and so do not, in general, coincide with the crystallographic coordinate axes. Hence, $p_{ijkl}' \neq p_{ij(kl)}$ whenever $k \neq l$ for triclinic crystals. This means that there are 18 more photoelastic tensor elements than previously believed for this crystal system.

For many crystals the optical anisotropy is sufficiently small that $p_{ij[kl]}$ will be negligible for them. However, for many strongly anisotropic crystals the rotational effect is sizable. These include calcite, sodium nitrate, cinnabar, rutile, lithium niobate, etc. For calcite at $\lambda = 6328$ Å, for instance, we predict on the basis of Eq. (8) that $(p_{4\bar{4}}' - p_{44}')/\frac{1}{2}(p_{4\bar{4}}' + p_{44}') = 98\%$ if the measured value[4] of p_{44} is used for the denominator.

It is interesting to note that we have also found rotational effects of acoustic waves in the theory[5] of acoustically induced, phase matched, optical harmonic generation.[6] In this experiment two input optic waves are mixed with an input acoustic wave to produce an outgoing optic wave at a frequency displaced from the optical second harmonic by the acoustic frequency. Here, as one could expect, the second-order optical constants, that is, the second-harmonic-generation tensor elements, are rotated by the acoustic wave. Rotational effects are, in fact, expected to contribute to acousto-optic interactions of all orders in optically anisotropic media when the acoustic wave posesses a shear component.

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