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[12]It corresponds to Eq. (4.7) of Ref. 1, except that the unknown pion-charge form factor was arbitrarily put equal to the nucleon isovector Dirac form factor. This did not create a serious problem since the pion-pole contribution is actually negligible for $q^2 \gtrsim 0.5$.

[13]We have also considered the effects of (a) nonzero neutron-charge form factor G_E^n, and (b) the vector-meson (ρ and ω) exchange. But these seem to be rather small (a few percent) and fail to explain the conspicuous change of slope. There is positive evidence that G_E^n does not vanish for small q^2 ($\lesssim 0.2$) (see Panofsky, Ref. 6), and its effect has been found to be appreciable by Gleeson, Gundzik, and Kuriyan, Ref. 5. We do not find it to be the case in our model. From the viewpoint of current algebra, the introduction of vector-meson diagrams is equivalent to the phenomenological modification of PCAC by R. Arnowitt, M. H. Friedman, and P. Nath, Phys. Letters 27B, 657 (1968).

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UNITARITY UPPER BOUND ON THE ABSORPTIVE PARTS OF ELASTIC-SCATTERING AMPLITUDES

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We derive a unitarity upper bound on the absorptive part $A(s,t)$ of the elastic-scattering amplitude in the physical region. In the diffraction peak region the bound on $A(s,t)/A(s,0)$ is a function only of the parameter $\sigma_{tot}^2(-t)/4\pi\sigma_{el}$. Assuming the diffraction scattering to be purely absorptive and spin independent we find the pp, $\bar{p}p$, and $\pi^\pm p$ data to lie a little below the theoretical upper bound. Further the experimental curve of $X(t)/X(0)$ vs $-4tX(0)/\sigma_{el}$, where $X(t) \equiv d\sigma/dt$, seems "universal."

(1) Introduction. –One of the important features of high-energy elastic scattering is the presence of the diffraction peak in the low-momentum-transfer region. We wish to investigate how far one can understand this feature in terms of restrictions coming from direct-channel unitarity alone without recourse to any specific model. We begin by recalling the unitarity upper bound on "diffraction peak width" obtained by MacDowell and Martin.[1] They prove that

$$\left[\frac{d}{dt}\ln A(s,t)\right]_{t=0} > \frac{1}{9}\left[\frac{\sigma_{tot}^2}{4\pi\sigma_{el}} - \frac{1}{k^2}\right], \tag{1.1}$$

where $A(s,t)$ is the absorptive part of the elastic-scattering amplitude for two spinless particles at c.m. energy $\sqrt{s}$ and momentum transfer squared t, k being the c.m. momentum. The σ_{tot} and σ_{el} are, respectively, the total and elastic cross sections. This bound is remarkably close to the observed experimental values when a comparison is made by assuming that the unpolarized cross sections in the diffraction-peak region are spin independent and purely absorptive. The MacDowell-Martin result leads us to hope that the differential cross section in the diffraction peak region might be understood similarly in terms of a unitarity upper bound on $A(s,t)$ in the physical region.

The main obstacle to obtaining good bounds on $A(s,t)$ in the physical region so far has been the oscil-

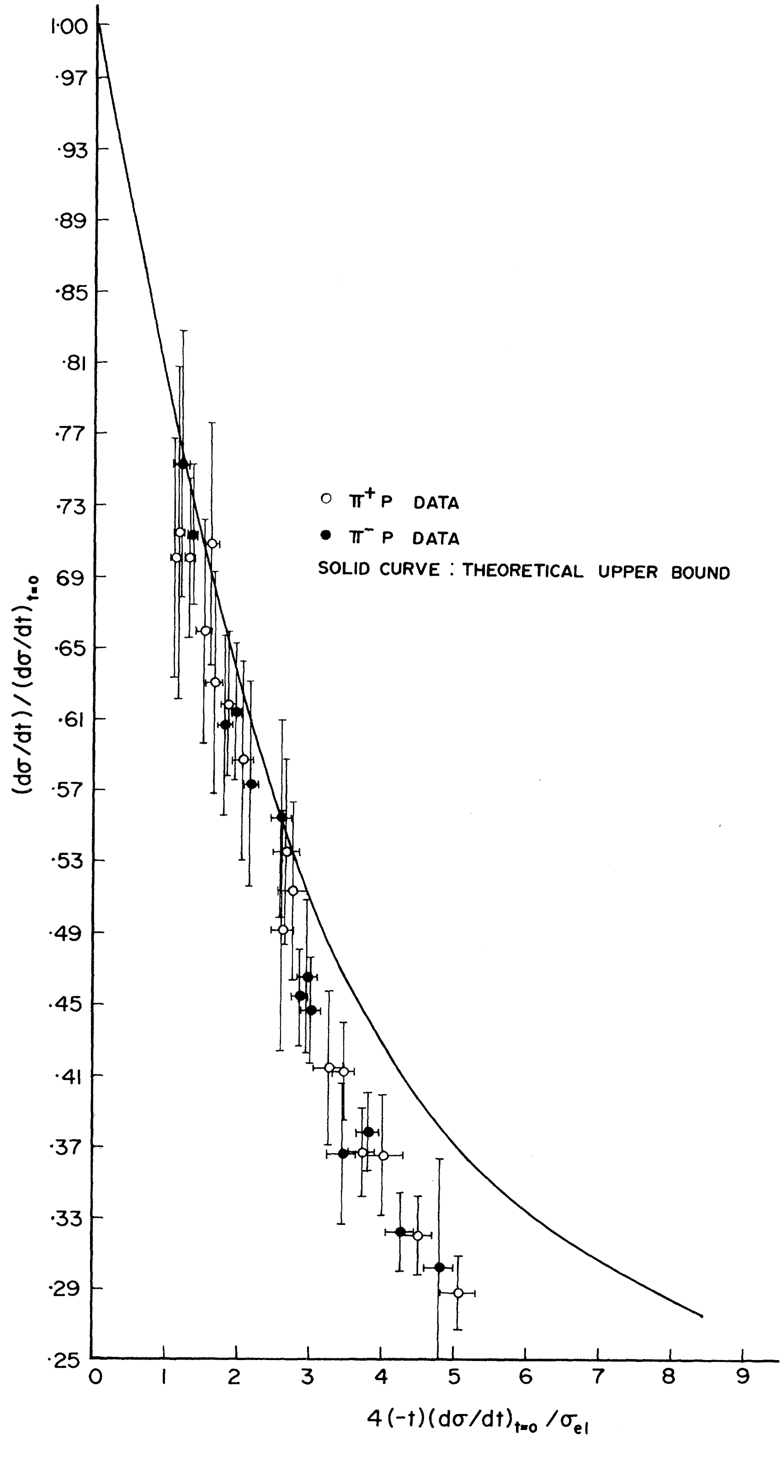

FIG. 1. Comparison between the theoretical upper bound on the curve of $[A(s,t)/A(s,0)]^2$ vs $(-t)\sigma_{tot}{}^2/4\pi\sigma_{el}$ and the experimental curve of $X(t)/X(0)$ vs $4(-t)X(0)\sigma_{el}$. Note that the quantities plotted in the theoretical and experimental curves are equal for purely absorptive, spin-independent scattering. We use the data of K. J. Foley et al., Phys. Rev. Letters 11, 503 (1963), on π^+p scattering at lab momenta 6.8, 8.8, 10.8, and 12.8 GeV/c and π^-p scattering at lab momenta 7.0, 8.9, 10.8, and 13.0 GeV/c. The pp and $\bar{p}p$ data in the same energy range also fall on the same experimental curve as the $\pi^\pm p$ data but have not been shown to avoid overcrowding.

latory behavior as a function of l of the Legendre polynomials $P_l(\cos\theta)$ occurring in the partial-wave expansion for $\pi > \theta > 0$. Martin,[2] for this reason, has worked with nonoscillatory functions which majorize $|P_l(\cos\theta)|$ and he thereby obtained an upper bound on $A(s,t)$ in terms of $\sigma_{\rm tot}$. This procedure entails loss of information. We have therefore tackled the oscillation problem frontally. Our procedure yields an improved upper bound on $A(s,t)$ in terms of $\sigma_{\rm tot}$ which we shall report elsewhere.

Here we report an upper bound on $A(s,t)$ which is in terms of $\sigma_{\rm tot}$ and $\sigma_{\rm el}$ and which is of practical interest. This upper bound is a function only of the parameter ρ given by

$$\rho = \sigma_{\rm tot}^2(-t)/4\pi\sigma_{\rm el} \tag{1.2}$$

in the diffraction-peak region. It has a particularly simple form for small values of ρ given by

$$\frac{A(s,t)}{A(s,0)} \leqslant \left[1-\frac{\rho}{9}+\frac{3}{8}\left(\frac{\rho}{9}\right)^2-\frac{21}{320}\left(\frac{\rho}{9}\right)^3+\cdots\right] \text{ for } 2.5 \geqslant \rho \geqslant 0. \tag{1.3}$$

We give explicit formulas and numerical values for the upper bound up to $\rho = 8.42$. We compare our theoretical upper bound with the experimental data on pp, $\bar{p}p$, π^+p, and π^-p scattering in the diffraction peak region by assuming that the unpolarized differential cross sections are (i) spin independent and (ii) purely absorptive. We then have

$$X(t)/X(0) = [A(s,t)/A(s,0)]^2 \tag{1.4}$$

where $X(t) \equiv d\sigma/dt$, and

$$4X(0)(-t)/\sigma_{\rm el} = \sigma_{\rm tot}^2(-t)/4\pi\sigma_{\rm el} = \rho. \tag{1.5}$$

We can therefore compare, under these assumptions, the experimental curve of $X(t)/X(0)$ vs $4(-t)\times X(0)/\sigma_{\rm el}$ with our theoretical upper bound for the curve of $[A(s,t)/A(s,0)]^2$ vs ρ. The experimental points lie only slightly below our theoretical curve, differing by less than 10% for ρ in the range 0 to 3 and by less than 25% for ρ in the range 3 to 5. We further notice that the experimental curve seems to be "universal" (Fig. 1).

We are intrigued by this universality feature exhibited by the experimental data and by its close agreement with our theoretical upper bound. It may be that the variable ρ has some deeper significance.

(2) Upper-bound theorem.—We give below the exact upper-bound theorem on $A(s,t)$ which, on evaluation in the diffraction-peak region, gives (1.3) and other results.

Theorem.—Let $(k^2/4\pi)(\sigma_{\rm tot}^2/\sigma_{\rm el}) \geqslant 1$ and $\pi > \theta > 0$ where θ is the scattering angle $[t = -2k^2(1-\cos\theta)]$. Then

$$\frac{A(s,t)}{A(s,0)} \leqslant U(s,t) = \frac{\sum_u (2l+1)P_l(\cos\theta)[P_l(\cos\theta)-A]}{\sum_u (2l+1)[P_l(\cos\theta)-A]}, \tag{2.1}$$

where A $(1 \geqslant A > 0)$ is determined by

$$\frac{k^2\sigma_{\rm tot}^2}{4\pi\sigma_{\rm el}} = \frac{\{\sum_u (2l+1)[P_l(\cos\theta)-A]\}^2}{\sum_u (2l+1)[P_l(\cos\theta)-A]^2}. \tag{2.2}$$

Here $\sum_u$ stands for summation over those l values which satisfy

$$P_l(\cos\theta) \geqslant A. \tag{2.3}$$

The proof of this theorem will be indicated later. Here we shall only make a few comments on it.

(1) We only need the physical-region partial-wave expansion of $A(s,t)$ given by

$$A(s,t) = \frac{\sqrt{s}}{k}\sum_{l=0}^{\infty}(2l+1)\,\mathrm{Im}a_l(s)P_l(\cos\theta) \tag{2.4}$$

and the positivity of the imaginary parts of the partial-wave amplitudes $[\text{i.e.}, \mathrm{Im}a_l(s) \geqslant 0]$ for proving the theorem.

(2) The upper bound $U(s,t)$ given above is achieved for the following set of $\mathrm{Im}a_l(s)$'s:

$$\mathrm{Im}a_l^{(0)}(s) = \alpha[P_l(\cos\theta)-A]\theta(P_l(\cos\theta)-A), \tag{2.5}$$

where $\alpha > 0$ and $\theta(x)$ is the step function of x and where α and A are chosen so as to reproduce the known value of σ_{tot} and σ_{el}, i.e.,

$$\sigma_{tot}(s) = (4\pi\alpha/k^2)\sum_u (2l+1)[P_l(\cos\theta) - A], \tag{2.6}$$

$$\sigma_{el}(s) = (4\pi\alpha^2/k^2)\sum_v (2l+1)[P_l(\cos\theta) - A]^2. \tag{2.7}$$

Obviously A given by (2.6) and (2.7) is the same as that given by (2.2). Further the positivity of α also follows from (2.6) and (2.7).

(3) Evaluation of the upper bound in the diffraction-peak region. —We are interested in studying the upper bound $U(s,t)$ in the diffraction-peak region. For $\pi > \theta > 0$, $P_l(\cos\theta)$ is an oscillating function of l with maxima points $l = L_i$ given by

$$\left[\frac{\partial}{\partial l} P_l(\cos\theta)\right]_{l=L_i} = 0, \quad \left[\frac{\partial^2}{\partial l^2} P_l(\cos\theta)\right]_{l=L_i} < 0.$$

Further let

$$A_0 = \max_i [P_{L_i}(\cos\theta)]. \tag{3.1}$$

Case (1). —Consider first the case with A satisfying

$$1 \geq A > A_0. \tag{3.2}$$

The summations $\sum_u$ in (2.1) and (2.2) now run over $L \geq l \geq 0$ where L is the largest integer such that $P_L(\cos\theta) \geq A$. We then have

$$U(s,t) = \sum_{l=0}^{L} (2l+1)P_l(\cos\theta)[P_l(\cos\theta) - A] / \sum_{l=0}^{L} (2l+1)[P_l(\cos\theta) - A], \tag{3.3}$$

provided that A is given by the equation

$$\frac{k^2 \sigma_{tot}^2}{4\pi\sigma_{el}} = \left\{\sum_{l=0}^{L} (2l+1)[P_l(\cos\theta) - A]\right\}^2 / \sum_{l=0}^{L} (2l+1)[P_l(\cos\theta) - A]^2. \tag{3.4}$$

The summation over l in Eqs. (3.3) and (3.4) can be done exactly by using

$$\sum_{l=0}^{L} (2l+1)P_l(\cos\theta) = P_{L+1}'(\cos\theta) + P_L'(\cos\theta),$$

$$\sum_{l=0}^{L} (2l+1)[P_l(\cos\theta)]^2 = (L+1)[P_L(\cos\theta)]^2 + \sin^2\theta[P_L'(\cos\theta)]^2. \tag{3.5}$$

In the diffraction-peak region s is large and $\theta \approx 0$ so that the Legendre polynomials can be approximated by Bessel functions and we obtain from (3.3), (3.4), and (3.5)

$$\frac{A(s,t)}{A(s,0)} \leq U(s,t) \underset{\substack{s\to\infty \\ \theta\approx 0}}{=} J_0(\eta) + (\eta^2/\rho)J_2(\eta), \tag{3.6}$$

where $\eta = (2L+1)\sin\frac{1}{2}\theta$, $\rho = (-t)\sigma_{tot}^2/4\pi\sigma_{el}$, and η is to be determined by the equation

$$\rho = [\eta J_2(\eta)]^2 / \{[J_1(\eta)]^2 - 2J_0(\eta)J_2(\eta)\}. \tag{3.7}$$

The condition $1 \geq A > A_0$ is seen to be equivalent to

$$2.5 \geq \rho \geq 0. \tag{3.8}$$

Using (3.6) and (3.7) and the expansion of Bessel functions in powers of η we obtain the simple result (1.3) quoted earlier in the introduction.

Case (2). —We now consider the remaining case, i.e., $A_0 \geq A > 0$, which is equivalent to considering value of ρ larger than 2.5. There is more than one piece in the l summation. Following an exactly similar procedure we obtain

$$\frac{A(s,t)}{A(s,0)} \underset{\substack{s\to\infty \\ \theta\approx 0}}{\leq} J_0(\mu_0) + \frac{1}{\rho}\sum_i [\mu_i^2 J_2(\mu_i) - \eta_i^2 J_2(\eta_i)], \tag{3.9}$$

where

$$0 = \eta_0 \leq \mu_0 \leq \eta_1 \leq \mu_1 \leq \eta_2 \leq \mu_2 \leq \cdots; \tag{3.10}$$

$$J_0(\eta_l) = J_0(\mu_l) = J_0(\mu_0) \tag{3.11}$$

Table I. Upper bounds on the imaginary part $A(s,t)$ in the diffraction-peak region as a function of $\rho=(-t)(\sigma_{tot}^2/4\pi\sigma_{el})$.

ρ	Upper bound on $A(s,t)/A(s,0)$	ρ	Upper bound on $A(s,t)/A(s,0)$
0	1.000	4.50	0.630
0.50	0.945	5.00	0.610
1.00	0.893	5.50	0.593
1.50	0.846	6.00	0.578
2.00	0.795	6.50	0.566
2.50	0.749	7.00	0.554
3.00	0.710	7.50	0.543
3.50	0.677	8.00	0.533
4.00	0.652	8.42	0.523

for all $i\neq 0$ and the η_i and μ_i are determined by

$$\rho=\frac{\{\sum_i[\mu_i^2J_2(\mu_i)-\eta_i^2J_2(\eta_i)]\}^2}{\sum_i\{[\mu_iJ_1(\mu_i)]^2-[\eta_iJ_1(\eta_i)]^2\}-2J_0(\mu_0)\{\sum_i[\mu_i^2J_2(\mu_i)-\eta_i^2J_2(\eta_i)]\}}. \tag{3.12}$$

The numerical values of the upper bound for $0\leqslant\rho\leqslant 8.42$ are given in Table I, since we felt that there is not much practical use for the upper bound for larger ρ values.

(4) Proof.—We now prove the theorem given in the Sec. (2) by "direct-subtraction" method. Let

$$A^{(0)}(s,t)=\frac{\sqrt{s}}{k}\sum_{l=0}^{\infty}(2l+1)\,\mathrm{Im}a_l^{(0)}(s)P_l(\cos\theta), \tag{4.1}$$

where the $\mathrm{Im}a_l^{(0)}(s)$ are given by (2.5)-(2.7). Consider

$$\Delta\equiv\frac{k}{\sqrt{s}}[A(s,\epsilon)-A^{(0)}(s,t)]=\sum_u(2l+1)[\mathrm{Im}a_l-\mathrm{Im}a_l^{(0)}]P_l(\cos\theta)+\sum_v(2l+1)\mathrm{Im}a_lP_l(\cos\theta), \tag{4.2}$$

where $\sum_v$ stands for summation over those l values which satisfy $A>P_l(\cos\theta)$. Eliminating $P_l(\cos\theta)$ in the first sum on the right-hand side of (4.2) in favor of $\mathrm{Im}a_l^{(0)}$ by using (2.5), we obtain

$$\Delta=\frac{1}{2\alpha}\sum_u(2l+1)[(\mathrm{Im}a_l)^2-(\mathrm{Im}a_l^{(0)})^2-(\mathrm{Im}a_l-\mathrm{Im}a_l^{(0)})^2]+A\sum_u(2l+1)(\mathrm{Im}a_l-\mathrm{Im}a_l^{(0)})$$
$$+\sum_v(2l+1)\mathrm{Im}a_lP_l(\cos\theta). \tag{4.3}$$

Now (2.6) and (2.7) lead to

$$\sum_{l=0}^{\infty}(2l+1)\mathrm{Im}a_l=\sum_u(2l+1)\mathrm{Im}a_l^{(0)} \tag{4.4}$$

and

$$\sum_{l=0}^{\infty}(2l+1)(\mathrm{Re}a_l)^2+\sum_{l=0}^{\infty}(2l+1)(\mathrm{Im}a_l)^2=\sum_v(2l+1)(\mathrm{Im}a_l^{(0)})^2. \tag{4.5}$$

Using (4.4) and (4.5) in (4.3) we obtain

$$\Delta=-\frac{1}{2\alpha}\{\sum_{l=0}^{\infty}(2l+1)(\mathrm{Re}a_l)^2+\sum_{l=0}^{\infty}(2l+1)(\mathrm{Im}a_l-\mathrm{Im}a_l^{(0)})^2-2\alpha\sum_v(2l+1)\mathrm{Im}a_l[P_l(\cos\theta)-A]\}\leqslant 0. \tag{4.6}$$

We next note that the right-hand side of (2.2) varies from 1 to ∞ as A goes from one to zero and further that the summation in the numerator of (2.2) becomes divergent for A negative. Therefore, for $k^2\sigma_{tot}^2/4\pi\sigma_{el}\geqslant 1$ one can always find an A in the interval (0,1) to solve Eq. (2.2). This completes the proof.

(5) Concluding remarks.—(i) The upper bound given in Sec. (2) is the best possible one as long as only the knowledge of σ_{tot}, σ_{el}, and positivity of $\mathrm{Im}a_l$ is used. Use of the additional information $\mathrm{Im}a_l\leqslant 1$ does not lead to any improvement in the diffraction-peak region because $\mathrm{Im}a_l^{(0)}\leqslant 1$ is automatically satisfied for the experimental values of σ_{tot} and σ_{el}.

(ii) Our method also allows us to establish a lower bound on $A(s,t)$ in terms of σ_{tot} and σ_{el}. We find that in the range $0 \leq \rho \leq 8.42$ the lower bound on $A(s,t)/A(s,0)$ is always negative and less in magnitude than the upper bound $U(s,t)$ varying monotonically from -0.162 to -0.138 as ρ varies from 0 to 11.5. Therefore in the range $0 \leq \rho \leq 8.42$ the upper bound on $[A(s,t)/A(s,0)]^2$ is simply $[U(s,t)]^2$.

Further details and other upper and lower bounds in the physical and unphysical regions will be reported in a later detailed paper.[3]

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COMMENTS ON THE LETTER "EVIDENCE OF QUARKS IN AIR-SHOWER CORES"*

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The "quark tracks" observed by McCusker and co-workers in cloud chambers can be explained by making reasonable assumptions about two processes: fluctuations in the number of droplets in a cloud-chamber track, and the relativistic rise of the ionization. However, before a firm conclusion can be drawn, more experimental data on the drop-count distribution of a large sample of tracks in the experiment are needed.

McCusker and collaborators have recently performed an ingenious experiment to search for quarks in the cores of large air showers.[1,2] Among 5×10^4 tracks in delayed-expansion cloud chambers, they have found five with about one-half the ionization of comparison tracks in the same or near-by pictures; they interpret these as being due to quarks with charge $\frac{2}{3}$. The quark flux computed on the basis of this experiment is in mild disagreement with the result of searches in terrestrial matter.[3,4] This disagreement can be explained away because the chemical behavior of quarks is not known. Nevertheless it has led us to re-examine the evidence presented by the Sydney group.

In estimating the probability of a few deviant cases in a large sample, one is faced with the difficulty that the result is very sensitive to the value of the assumed standard deviation of the parameter measured. Only an experimental determination of the frequency of tracks as a function of drop count will permit a definitive distinction between a subsidiary peak at low ionization and the low-ionization tail of a broad peak. Meanwhile, consideration of what is known at present about drop-count statistics and the increase of ionization beyond the minimum suggests that the observed effect is not necessarily due to particles of reduced charge.

The increase in ionization beyond the minimum is well established[5]; the corresponding increase in drop count in a cloud chamber has been observed in many gases[6]; in argon it reaches 20% at $\gamma = 20$-40 and 40% at $\gamma = 100$-400 (γ is the energy in units of the rest energy). Shower particles rarely appear at minimum ionization in a cloud chamber.[7] In an air shower of 10^6 charged particles at sea level, typical[8] electron energies near the core are 0.5 GeV ($\gamma = 1000$); the muons have average energy exceeding 5 GeV ($\gamma = 50$). Under lead the electron energies are lower, but it is safe even here to assume that the average ionization is at least $1.2\times I_0$, where I_0 is the minimum ionization. To be conservative, we shall assume for the following estimates that particles with energies corresponding to ionization between I_0 and $1.3\times I_0$ are present, and that the number of particles is uniformly distributed over this range. Then the average ionization of the comparison tracks is $1.15\times I_0$ and the expected ionization of a charge $\frac{2}{3}$ quark at its minimum is $(4/9)/1.15 = 0.39$ times the average of the comparison tracks.

The fluctuation in the number, N_d, of droplets in cloud-chamber tracks of particles of fixed charge and velocity is not given by $(N_d)^{1/2}$, but is considerably larger.[9-11] Three processes are involved: the primary ionization by the fast particle; the secondary ionization by ejected electrons; and the formation of photographable drops on diffused ions. The primary ionization is indeed Poissonian, but the mean number involved[12]