$-E_n|$ at least several times the level spacing. The level spacing is $O(1/L)$. In the periodic case, $\int\varphi_m^*(d\varphi_n/dx)dx$ would be replaced by $k\delta_{k,k'}$. The range of k over which the $\delta_{k,k'}$ would be broadened in the aperiodic case would be $1/L_e$ where L_e is the phase-coherence length or mean free path. For the validity of (19) and (20), then, we require that $L_e \ll L$, which is one of the usual criteria for the validity of macroscopic transport theory. In the regime $L_e \ll L$, the quantity $|p_{\mathrm{ext}}|$ can be expected to depend linearly on $(L_e/L)^{1/2}$ (since L_e is a phase-coherence length[13,14]) and to be exponentially small (since L_e is also the localization length for the amplitude of the wave function). Consequently,

$$|p_{\mathrm{ext}}| \sim mv(L_e/L)^{1/2}e^{-L/L_e} \tag{21}$$

for E inside the "allowed zones," where v is a typical band velocity for energy E, and

$$\mu(E) \cong \frac{2\pi e\hbar}{kT}L_e v^2 n(E)e^{-L/L_e} \tag{22}$$

for E inside the "allowed zones," where $n(E)$ is the density of states per unit length. Thus μ is L dependent and very small in the regime $L \gg L_e$ which in one dimension corresponds to ordinary transport. The difference between one and higher dimensions occurs because it is only in one dimension that phase incoherence leads to localization of what would be extended states in higher dimensionality.

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DEVIATIONS FROM MATTHIESSEN'S RULE IN METALS WITH MAGNETIC IMPURITIES*

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It is shown that resonant scattering of the conduction electrons in a metal, when combined with ordinary impurity scattering and phonon scattering, leads to significant deviations from Matthiessen's rule for the total resistivity. The implication for the analysis of the observed resistivity of metals with magnetic impurities is discussed.

In this note we present the results of a variational calculation of the resistivity of a metal, in which the electrons suffer an energy-dependent scattering from impurities in addition to ordinary impurity scattering and scattering against phonons. Such a situation occurs for example in metals containing magnetic impurities. Experimentally, the contribution of the magnetic impurities to the resistivity of the metal is determined[1] from the difference between the measured resistivity of the impure sample and that of the pure material. Characteristically this difference drops from a maximum at zero temperature to some constant value at higher temperatures, in which case the constant term is subtracted off as due to ordinary potential scattering. The remainder is attributed to the effect of the magnetic impurities via conduction-electron scattering involving the spin of the impurity.

This simple subtraction procedure isolates the resistivity due to the magnetic impurities alone only if Matthiessen's rule holds, that is, only if the sum of the individual scattering rates (i.e., the rates that characterize each scattering mechanism considered separately) equals the total scattering rate in the presence of the several scattering mechanisms.

Deviations from this rule are known both theoretically[2] and experimentally.[3] In the present context we are concerned with deviations that occur because the individual scattering mechanisms, when considered separately, give rise to electron distribution functions that differ in their dependence on energy or direction in momentum space. We disregard, however, such deviations as are due to the influence of the impurities on the phonon spectrum or the electronic states of the pure metal, since presumably these become negligible in sufficiently dilute systems.

Several authors[4] have derived effective energy-dependent relaxation times for the scattering of electrons off magnetic impurities but have, so far, failed to account satisfactorily for the temperature dependence of the resistivity attributed to magnetic impurities. In the absence of an adequate theory we make the simple assumption that magnetic impurities may be treated as resonant scatterers at the Fermi energy μ with width Γ. We can then characterize the scattering by an energy-dependent relaxation time $\tau_{\mathrm{res}}(\epsilon)$,

$$\frac{1}{\tau_{\mathrm{res}}(\epsilon)} = \frac{1}{\tau_{\mathrm{res}}{}^0}\frac{\Gamma^2}{(\epsilon-\mu)^2+\Gamma^2}, \tag{1}$$

where the constant $1/\tau_{\mathrm{res}}{}^0$ is proportional to the concentration of magnetic impurities.

We use (1) to demonstrate how a strongly energy-dependent scattering may interfere with the scattering due to ordinary impurities and phonons to result in a pronounced nonadditivity of the individual rates.[5] The qualitative features of the result can be expected to remain valid whenever the scattering off the magnetic impurities has an appreciable energy dependence on the scale of $k_{\mathrm{B}}T$, even if not of the form (1).

The interference effects are simply demonstrated when only the two kinds of impurity scattering are present.[5a] The Boltzmann equation for the electron distribution function $f_{\vec{k}}$ in the presence of a (unit) electric field $\hat{E}$ is then

$$\begin{aligned} X_{\vec{k}} &\equiv (\partial f^0/\partial\epsilon)\vec{v}_{\vec{k}}\cdot\hat{E}e \\ &= -(\tau_{\mathrm{res}}{}^{-1}+\tau_{\mathrm{imp}}{}^{-1})(\partial f^0/\partial\epsilon)\psi_{\vec{k}}, \end{aligned} \tag{2}$$

where the electron of charge e and velocity $\vec{v}_{\vec{k}}$ has an equilibrium distribution function f^0. The relaxation time due to ordinary impurity scattering is τ_{imp}, and $\psi_{\vec{k}}$ is defined by

$$f_{\vec{k}} - f^0 = (\partial f^0/\partial\epsilon)\psi_{\vec{k}}. \tag{3}$$

With the notation $t=(\epsilon-\mu)/k_{\mathrm{B}}T$ and $\gamma=\Gamma/k_{\mathrm{B}}T$, the solution of (2) is

$$\psi_{\vec{k}} \equiv -\psi(t)\vec{v}_{\vec{k}}\cdot\hat{E}e, \tag{4}$$

$$= -\left(\frac{1/\tau_{\mathrm{res}}{}^0}{1+t^2/\gamma^2}+\frac{1}{\tau_{\mathrm{imp}}}\right)^{-1}\vec{v}_{\vec{k}}\cdot\hat{E}e. \tag{4a}$$

The conductivity σ is in general given by

$$\sigma = \sum_{\vec{k}} e\vec{v}_{\vec{k}}\cdot\hat{E}(\partial f^0/\partial\epsilon)\psi_{\vec{k}} = \sum_{\vec{k}} X_{\vec{k}}\psi_{\vec{k}}. \tag{5}$$

With (4a) in (5), the resistivity $\rho=\sigma^{-1}$ becomes

$$\begin{aligned} &\rho/(m/ne^2) \\ &\quad= (\tau_{\mathrm{res}}{}^0)^{-1}F(\gamma,\tau_{\mathrm{imp}}/\tau_{\mathrm{res}}{}^0)+(\tau_{\mathrm{imp}})^{-1}, \end{aligned} \tag{6}$$

where

$$F(\gamma,x) = \frac{g(\gamma')}{1+x[1-g(\gamma')]}, \tag{6a}$$

and

$$g(\gamma) = (\gamma/2\pi)\psi'(\tfrac{1}{2}+\gamma/2\pi), \quad \gamma'=\gamma(1+x)^{1/2}. \tag{6b}$$

Here ψ' is the trigamma function.

We wish to compare (6) with the resistivity one obtains by assuming the validity of Matthiessen's rule. When there is only ordinary impurity scattering present, the solution of (2) has the form (4) with $\psi_{\mathrm{imp}}(t)=\tau_{\mathrm{imp}}$, which gives the resistivity

$$\rho_{\mathrm{imp}} = m/ne^2\tau_{\mathrm{imp}}. \tag{7}$$

With only resonant scattering we get, similarly,

$$\psi_{\mathrm{res}}(t) = \tau_{\mathrm{res}}{}^0(1+t^2/\gamma^2) \tag{8}$$

and the resistivity

$$\rho_{\mathrm{res}} = (m/ne^2\tau_{\mathrm{res}}{}^0)[1+(\pi^2/3)(k_{\mathrm{B}}T/\Gamma)^2]^{-1}. \tag{9}$$

We define the deviation from Matthiessen's rule as

$$\Delta = \rho-(\rho_{\mathrm{imp}}+\rho_{\mathrm{res}}). \tag{10}$$

At low temperatures, $T \ll \Gamma/k_{\mathrm{B}}$ (i.e., $\gamma \gg 1$), the deviation Δ is negligible compared to ρ_{res} because ψ_{res} in that limit is nearly constant over the range of $k_{\mathrm{B}}T$ ($|t| \lesssim 1$) and hence very similar to the (constant) ψ_{imp}.

At high temperatures ($\gamma \ll 1$), $\psi_{\mathrm{res}}(t)$ is very different from ψ_{imp}. In this limit Δ is linear in $\gamma(=\Gamma/k_{\mathrm{B}}T)$ and hence dominates ρ_{res} which, according to (9), is quadratic in γ. If Γ is independent of temperature, Δ therefore decreases like T^{-1} for large T in contrast to the T^{-2} behavior of ρ_{res} as shown in Fig. 1.

Significant deviations from Matthiessen's rule persist in the presence of electron-phonon scattering. This additional mechanism can be in-

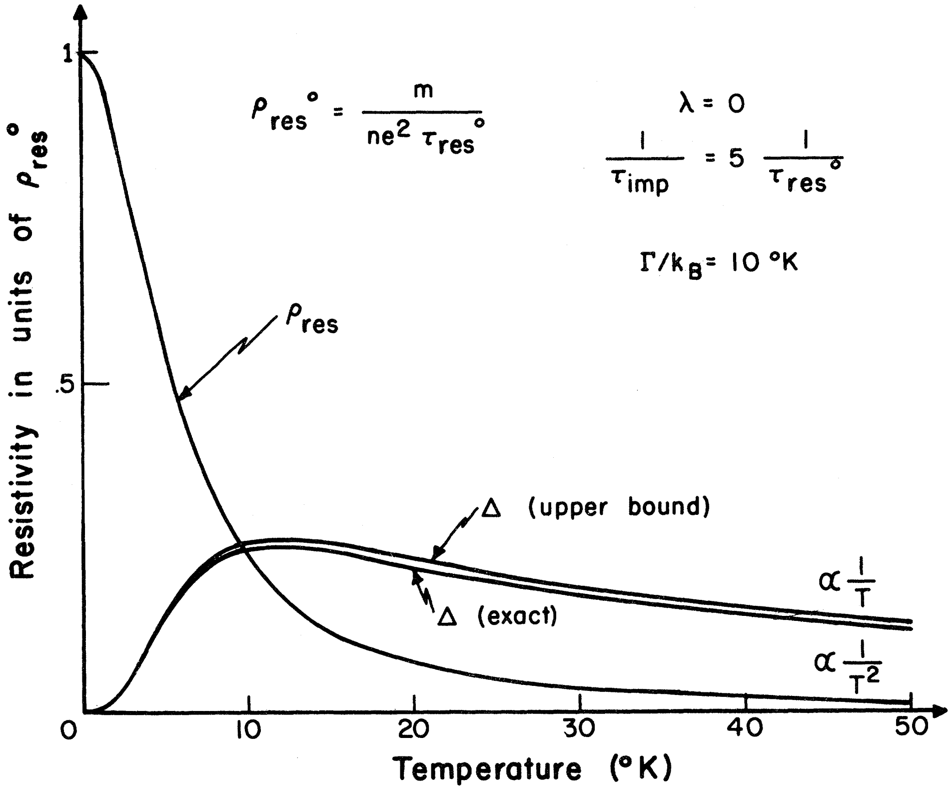

FIG. 1. Characteristic temperature dependence of the resistivity ρ_{res} [Eq. (9)] and the (exact) deviation from Matthiessen's rule Δ obtained from (6), (7), (9), and (10). The exact Δ is compared with the upper bound on Δ obtained using (13). The values of the various parameters are shown in the figure.

cluded in the Boltzmann equation by adding

$$(1/k_BT)\sum_{\vec{k}'}V(\vec{k},\vec{k}')(\psi_{\vec{k}}-\psi_{\vec{k}'})$$

to the right-hand side of (2). We treat the electron-phonon collision operator $V(\vec{k},\vec{k}')$ according to a scheme[6] which allows us to take into account the effects of band structure and the details of the phonon spectrum. For the sake of simplicity, however, we have here assumed a spherical Fermi surface for the electrons and a Debye spectrum for the phonons which are taken to be in equilibrium. The collision operator $V(\vec{k},\vec{k}')$ is (cf. Wilson[7])

$$V(\vec{k},\vec{k}') = \frac{2\pi}{\hbar}|g_{\vec{k}-\vec{k}'}|^2 \frac{f_{\vec{k}}{}^0 f_{\vec{k}'}{}^0}{|e^{-\epsilon_{\vec{k}}/k_BT} - e^{-\epsilon_{\vec{k}'}/k_BT}|}[\delta(\epsilon_{\vec{k}}-\epsilon_{\vec{k}'}-\omega_{\vec{k}-\vec{k}'}) + \delta(\epsilon_{\vec{k}}-\epsilon_{\vec{k}'}+\omega_{\vec{k}-\vec{k}'})]. \tag{11}$$

In (11) $g_{\vec{k}-\vec{k}'}$ is the matrix element for the scattering of an electron from $\vec{k}$ to $\vec{k}'$ through the emission or absorption of a phonon with energy $\omega_{\vec{k}-\vec{k}'}$. We use a simplified form for the matrix element that emphasizes its low-temperature properties, namely

$$|g_{\vec{q}}|^2 = \lambda\omega_{\vec{q}}/2N_0, \tag{12}$$

where λ is a characteristic constant of order unity and $2N_0$ is the density of states at the Fermi surface.

With the phonons included, (2) is too complicated to solve exactly. Instead, we use the upper bound[8] on the resistivity ρ,

$$\rho \leqslant \left[\sum_{\vec{k}}\left(-\frac{\partial f^0}{\partial \epsilon}\right)\left(\frac{1}{\tau_{res}} + \frac{1}{\tau_{imp}}\right)U_{\vec{k}}{}^2 + \frac{1}{2k_BT}\sum_{\vec{k},\vec{k}'}V(\vec{k},\vec{k}')(U_{\vec{k}}-U_{\vec{k}'})^2\right](\sum_{\vec{k}}X_{\vec{k}}U_{\vec{k}})^{-2}. \tag{13}$$

The trial function $U_{\vec{k}}$ has the form $U_{\vec{k}} = U(t)\vec{v}_{\vec{k}}\cdot\hat{E}$ where the energy-dependent part is varied to get the best upper bound.

A favorable choice of $U(t)$ is obtained by considering the exact solutions $\psi(t)$ belonging to each of the three scattering mechanisms separately. The separate solutions $\psi_{\mathrm{imp}}(t) = \mathrm{const}$ and $\psi_{\mathrm{res}}(t) \propto (1+t^2/\gamma^2)$ suggest the variable linear combination[9]

$$U(t) = p + c(1-p)(1+t^2/\gamma^2). \tag{14}$$

The constant c in (14) is chosen to make the denominator of (13) independent of the variational parameter p. When $\tau_{\mathrm{imp}}^{-1} = \tau_{\mathrm{res}}^{-1} = 0$, we obtain essentially the exact phonon resistivity[2] ρ_{ph} from (14) and (13), the bulk of the correction to the Bloch-Grüneisen formula [obtained with $U(t) = 1$] being accounted for by a (small) t^2 term in (14). In agreement with Sondheimer[2] we find the deviations from Matthiessen's rule due to the combined influence of ordinary impurities and phonons to be very small, which simply reflects the fact that the exact solution ψ_{ph} with only phonons present is nearly constant over the energy range of interest and hence similar to ψ_{imp}.

When the resonant scattering is included along with the two other scattering mechanisms, substantial deviations from Matthiessen's rule result. A typical case is shown in Fig. 2 where Δ [given by the obvious generalization of (10)], ρ_{res}, and $\rho_{res}+\Delta$ have been plotted against temperature. The parameters $\tau_{res}{}^0$, τ_{imp}, Γ, λ, and the Debye temperature Θ_{D} are chosen to conform with the experimental conditions for Cu:Cr.[1] We have set Γ/k_{B} equal to the estimated Kondo temperature and adjusted λ to get the right room-temperature resistivity for Cu. The values of $\tau_{\mathrm{res}}{}^0$ and τ_{imp} reproduce the "step height"[1] and total residual resistivity. The deviation Δ peaks around $T \sim \Gamma/k_{\mathrm{B}}$ and completely dominates ρ_{res} at high temperatures, where it falls off as T^{-1}. Deviations of the same order of magnitude occur if Γ, instead of being constant, decreases with temperature. To test the efficiency of the trial function (14) in (13), we have computed (13) in the absence of phonon scattering and it compares well with the exact result (6) as can be seen in Fig. 1.

We can summarize the implications of the interference effects for the experimental analysis of the resistivity due to magnetic impurities as follows: The resistivity $(\rho_{\mathrm{res}}+\Delta)$ one obtains by subtracting $\rho_{imp}+\rho_{ph}$ from the total resistivity ρ differs substantially from ρ_{res} when $T \gtrsim \Gamma/k_{\mathrm{B}}$ (Fig. 2). The combined effect of the three scattering mechanisms is to cause $\rho_{res}+\Delta$ to fall off over a wider range of temperature than ρ_{res} itself does. The magnitude of the effect is biggest when the phonons or the ordinary impurities dominate the scattering relative to the resonant

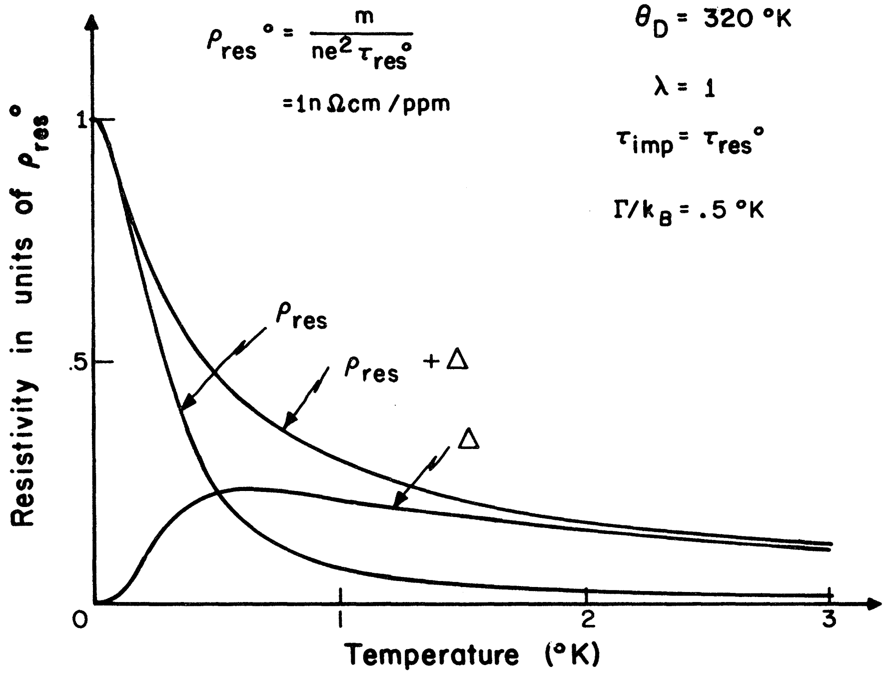

FIG. 2. Typical results for ρ_{res} and Δ in the case where all three scattering mechanisms are present. The parameters shown in the figure have been chosen to mimic the Cu:Cr study of Ref. 1 as discussed in the text.

scattering. We expect a similar conclusion to hold whenever the magnetic-impurity scattering has an appreciable energy dependence within the range of $k_B T$. Any comparison between experiment and theory, although the latter may invoke a different effective relaxation time than (1), suffers therefore from the considerable complication that the magnitude of the mutual interference between the various scattering mechanisms would have to be determined from the Boltzmann equation, in order to assess what part of the measured total resistivity is due to magnetic impurities considered separately and what part is due to interference.

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REMARKS ON THE RENORMALIZATION OF LOCAL SPIN FLUCTUATIONS IN METALS

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By a perturbation-theoretic calculation, we find that vertex corrections make an important contribution to the renormalized spin-fluctuation temperature in dilute alloys, in the limit of strong local enhancement.

There has been a considerable amount of interest in locally enhanced spin fluctuations in nearly magnetic dilute alloys. Qualitative descriptions of this phenomenon have appeared that base the discussion on a random-phase–approximation (RPA) calculation of the dynamic susceptibility of the inhomogeneous system.[1] Attempts to treat this problem more rigorously have appeared,[2,3] because the RPA breaks down when the local enhancement is very strong. In these theories, one obtains a self-consistent renormalized one-electron propagator by including self-energy graphs known to be important in the region of strong local enhancement. But vertex corrections have not been included in these studies. In this note we point out that in the limit of strong local enhancement, vertex corrections to the $T=0$ static susceptibility χ are equally or more important (in a sense defined below) than self-energy corrections. We feel that inclusion of vertex corrections in the theory may remove a number of difficulties with recent theories that have been