to the left of $\mathrm{Re}\,l=-\frac{1}{2}$.

[9]In general, we expect $g^2(l)$ to be a meromorphic function of $m(l)$. Even powers of $m(l)$ yield a fixed cut in the spin-flip amplitude, while odd powers give a fixed cut in the nonflip amplitude. Nucleon exchange in backward π^+p scattering may be qualitatively fitted with only even powers of $m(l)$ in $g^2(l)$.

[10]The contribution of the Feynman diagram for the exchange of an unnatural parity resonance has the form of (1) with $m(l)$ replaced by $-m(l)$. $\mathfrak{M}(m(l))+\mathfrak{M}(-m(l))$ has no odd powers of $m(l)$ and hence no cut in l.

[11]R. L. Sugar and J. D. Sullivan, Phys. Rev. 166, 1515 (1968).

[12]See D. L. Steele, thesis, University of Illinois, 1969 (unpublished).

[13]If $g^2(\alpha_0)=0$, there would be no infinity in (7) and no auxiliary pole $\alpha_2(u)$. However, the Regge-pole contribution to the nonflip part of $\mathfrak{M}$ at $u=0$ would be of order $b(u)$, which by assumption is small. Nucleon-exchange data in backward π^+p scattering shows no indication of any dip at $u=0$. (See also Ref. 9.)

[14]Note that α_1 and α_2 collide at $u=0$ and become complex for $u<0$. If there is only one moving pole (see Ref. 13), then α does not become complex. Also note that $\alpha_{1,2}(0)$ no longer coincide with the branch point α_0.

SOFT-PHOTON THEOREMS AND RADIATIVE K_{l3} DECAYS*

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(Received 12 December 1969)

We apply the soft-photon theorems of Low, Adler and Dothan, and Burnett and Kroll to the radiative decays $K^-\to\pi^0l^-\bar{\nu}\gamma$ and $\bar{K}^0\to\pi^+l^-\bar{\nu}\gamma$ ($l=e$ or μ) to obtain the leading terms in the respective matrix elements. Numerical results for the photon spectra and for the decay rate (as a function of the minimum photon energy) are given in terms of the conventional K_{l3} parameters $f_+(0)$, ξ, λ_+, and λ_-.

In this Letter we present the results of an extensive theoretical investigation of radiative K_{l3} decays. Because the ordinary K_{l3} decays have been studied in great detail, both experimentally and theoretically, the radiative modes provide a unique opportunity to check the predictions of soft-photon theorems, in particular the presence of derivative terms, which are not present in two-body decays and very difficult to observe in scattering processes due to the lack of a simple theory of the elastic-scattering matrix elements. A few of these radiative events have already been seen[1] and thus our results are of immediate interest to those phyicists working in this area. With slight modifications, experiments now in progress[1] could be designed to examine radiative K_{l3} events and check our theoretical predictions. We give all our results in terms of standard K_{l3} parameters $f_+(0)$, ξ, λ_+, and λ_-. For full details of the calculations, we refer the reader to a previous paper[2] and to another to be submitted for publication.[3]

Let us write down the relevant K_{l3} matrix element to establish our notation. Assuming the $|\Delta\vec{I}|=\frac{1}{2}$ rule, $V-A$ theory, and μ-e universality, we obtain the T-matrix element for $\bar{K}^0(P)\to\pi^+(Q)+l^-(p)+\bar{\nu}(q)$ (or $K^-\to\pi^0l^-\bar{\nu}$):

$$T(K\to\pi l\nu)=\bar{u}(p)[f_+(t)i\gamma\cdot(P+Q)+f_-(t)i\gamma\cdot(P-Q)](1+\gamma_5)v(q), \quad (1)$$

where $t=-(P-Q)^2$. In the SU(3) limit $f_-(0)=0$ and $f_+(0)=1/\sqrt{2}$ or 1 for charged or neutral K decays, respectively, and are real as we neglect CP-nonconserving effects. The decay rates are

$$\Gamma(K^-\to\pi^0e^-\bar{\nu})=4\Gamma_0\sin^2\theta f_+^{\,2}(0)(1.1826+4.3725\lambda_+)\times10^{-2}, \quad (2)$$

$$\Gamma(\bar{K}^0\to\pi^+e^-\bar{\nu})=4\Gamma_0\sin^2\theta f_+^{\,2}(0)(1.1977+4.1396\lambda_+)\times10^{-2}, \quad (3)$$

$$\Gamma(K^-\to\pi^0\mu^-\bar{\nu})=4\Gamma_0\sin^2\theta f_+^{\,2}(0)[0.7636+4.4925\lambda_++0.0227\xi^2+0.1992\xi^2\lambda_-+0.1495\xi+0.5622\xi(\lambda_++\lambda_-)]\times10^{-2}, \quad (4)$$

$$\Gamma(\bar{K}^0\to\pi^+\mu^-\bar{\nu})=4\Gamma_0\sin^2\theta f_+^{\,2}(0)[0.7728+4.2474\lambda_++0.0223\xi^2+0.1828\xi^2\lambda_-+0.1492\xi+0.5234\xi(\lambda_++\lambda_-)]\times10^{-2}, \quad (5)$$

where $\xi=f_-(0)/f_+(0)$, θ is the Cabibbo angle, $\Gamma_0=G^2\bar{M}^5/64\pi^3=3.118\times10^9\ \mathrm{sec}^{-1}$ with $G=1.435\times10^{-49}$ erg cm^3 the Fermi constant obtained from muon decay and with $\bar{M}=\frac{1}{2}(M_{K^0}+M_{K^-})$, and where the parameters $\lambda_\pm$ are defined by $f_\pm(t)=f_\pm(0)(1+\lambda_\pm t/m_\pi^2)$. It should be emphasized that we have used the exact masses in all calculations and the factor $\bar{M}^5$ was divided out explicitly to make the numbers in brackets dimensionless and to express them in the same units for all processes.[4]

The matrix elements for the radiative processes can be obtained using the general work of Low[5] and of Adler and Dothan.[6] One takes the contributions of diagrams corresponding to radiation from external charged lines, expands them in the photon energy k about $k=0$, and adds terms necessary to give a gauge-invariant result. The Low theorem states that this procedure gives, in a model-independent way, all contributions to the matrix element which are of order k^{-1} and k^0. The order k terms, i.e., the structure-dependent terms, correspond to radiation from internal lines and must be obtained from a particular model. The resulting T-matrix elements are

$$T(K^-\to\pi^0l^-\bar{\nu}\gamma)=\bar{u}(p)\left(\frac{\epsilon\cdot p}{k\cdot p}-\frac{\epsilon\cdot P}{k\cdot P}+\frac{\gamma\cdot\epsilon\gamma\cdot k}{2k\cdot p}\right)[2f_+(t)i\gamma\cdot Q-mf_1(t)](1+\gamma_5)v(q)$$
$$-2\left(\epsilon\cdot Q-k\cdot Q\frac{\epsilon\cdot P}{k\cdot P}\right)\bar{u}(p)\left[2\frac{\partial}{\partial t}f_+(t)i\gamma\cdot Q-m\frac{\partial}{\partial t}f_1(t)\right](1+\gamma_5)v(q)$$
$$+\text{structure-dependent terms of } O(k), \qquad (6)$$

$$T(\bar{K}^0\to\pi^+l^-\bar{\nu}\gamma)=\bar{u}(p)\left(\frac{\epsilon\cdot p}{k\cdot p}-\frac{\epsilon\cdot Q}{k\cdot Q}+\frac{\gamma\cdot\epsilon\gamma\cdot k}{2k\cdot p}\right)[2f_+(t)i\gamma\cdot P+mf_2(t)](1+\gamma_5)v(q)$$
$$+2\left(\epsilon\cdot P-k\cdot P\frac{\epsilon\cdot Q}{k\cdot Q}\right)\bar{u}(p)\left[2\frac{\partial}{\partial t}f_+(t)i\gamma\cdot P+m\frac{\partial}{\partial t}f_2(t)\right](1+\gamma_5)v(q)$$
$$+\text{structure-dependent terms of } O(k), \qquad (7)$$

where ϵ_μ is the photon polarization vector, m is the lepton mass, and $f_1=f_++f_-$, $f_2=f_+-f_-$. Note that these amplitudes contain an infrared-divergent term proportional to the nonradiative amplitude and two terms of order k^0, one of which involves derivatives of K_{l3} form factors.

The squares of these matrix elements (summed over the photon polarization) involve terms of order k^{-2} and k^{-1} which can be obtained immediately using the theorem of Burnett and Knoll[7] and terms of order k^0 which have to be calculated by usual methods.[8] In the square of the matrix element, we have consistently kept all terms of order k^{-2}, k^{-1}, and k^0 except those order-k^0 terms coming from interference between the structure-dependent and infrared-divergent parts of the matrix element and except for all terms of second order in $\lambda_\pm$. The terms which were neglected are very small, as discussed below.

The terms retained, after one momentum has been eliminated using four-momentum conservation, number approximately 250 in each decay and must be integrated over the three remaining final momenta. We used invariant integration techniques to do all of the integrations algebraically except for the final two which were done using Gaussian quadratures. The last integration in both cases was the photon energy so that we could make the proper cutoff $E_\gamma>E_{\min}$. Our final results for the radiative rate can be expressed in the form

$$\Gamma(K\to\pi l\bar{\nu}\gamma,\ E_\gamma>E_{\min})=\Gamma_0\sin^2\theta f_+^2(0)(A_1+A_2\lambda_++A_3\xi^2+A_4\xi^2\lambda_-+A_5\xi+A_6\xi\lambda_++A_7\xi\lambda_-). \qquad (8)$$

The coefficients A_1 to A_7 are given in Table I for the various modes and for several values of the minimum photon energy $E_{\min}$. For the electron mode A_3 to A_7 are essentially zero as they are proportional to the lepton mass.

For specific values of the K_{l3} parameters ξ and $\lambda_\pm$, we can obtain branching ratios; e.g., assuming $\lambda_+=0.029$, $\lambda_-=0$ for K^- and $\lambda_+=0.019$, $\lambda_-=0$ for $\bar{K}^0$,[9] we obtain, for $\xi=0$, -1, respectively,

$$\frac{\Gamma(K^-\to\pi^0e^-\bar{\nu}\gamma,\ E_\gamma>30\text{ MeV})}{\Gamma(K^-\to\pi^0e^-\bar{\nu})}=2.04\times10^{-2},$$

$$\frac{\Gamma(K^-\to\pi^0\mu^-\bar{\nu}\gamma,\ E_\gamma>30\text{ MeV})}{\Gamma(K^-\to\pi^0\mu^-\bar{\nu})}=0.739\times10^{-3},\quad 0.780\times10^{-3},$$

Table I. Numerical coefficients A_1 to A_7 in Eq. (8) for the rate for the radiative process $K \to \pi l \nu \gamma$ as a function of the minimum photon energy $E_{\min}$.

	$K^- \to \pi^0 \mu^- \bar{\nu}\gamma$			$\bar{K}^0 \to \pi^+ \mu^- \bar{\nu}\gamma$		
	E_{min}=20 MeV	E_{min}=30 MeV	E_{min}=40 MeV	E_{min}=20 MeV	E_{min}=30 MeV	E_{min}=40 MeV
A_1	3.206×10^{-5}	2.038×10^{-5}	1.354×10^{-5}	1.050×10^{-4}	6.786×10^{-5}	4.558×10^{-5}
A_2	2.852×10^{-4}	2.084×10^{-4}	1.578×10^{-4}	5.007×10^{-4}	3.264×10^{-4}	2.224×10^{-4}
A_3	1.323×10^{-6}	8.799×10^{-7}	6.107×10^{-7}	2.277×10^{-6}	1.434×10^{-6}	9.443×10^{-7}
A_4	8.620×10^{-6}	5.252×10^{-6}	3.353×10^{-6}	1.661×10^{-5}	1.052×10^{-5}	7.001×10^{-6}
A_5	5.797×10^{-6}	3.586×10^{-6}	2.318×10^{-6}	1.304×10^{-5}	7.688×10^{-6}	4.672×10^{-6}
A_6	1.748×10^{-5}	1.006×10^{-5}	6.079×10^{-6}	3.861×10^{-5}	2.100×10^{-5}	1.230×10^{-5}
A_7	1.624×10^{-5}	8.959×10^{-6}	5.138×10^{-6}	3.413×10^{-5}	1.863×10^{-5}	1.028×10^{-5}
	$K^- \to \pi^0 e^- \bar{\nu}\gamma$			$\bar{K}^0 \to \pi^+ e^- \bar{\nu}\gamma$		
	E_{min}=20 MeV	E_{min}=30 MeV	E_{min}=40 MeV	E_{min}=20 MeV	E_{min}=30 MeV	E_{min}=40 MeV
A_1	1.325×10^{-3}	9.616×10^{-4}	7.265×10^{-4}	1.573×10^{-3}	1.138×10^{-3}	8.567×10^{-4}
A_2	5.253×10^{-3}	3.752×10^{-3}	2.761×10^{-3}	5.734×10^{-3}	4.225×10^{-3}	3.239×10^{-3}

$$\frac{\Gamma(\bar{K}^0 \to \pi^+ e^- \bar{\nu}\gamma,\ E_\gamma > 30\ \text{MeV}}{\Gamma(\bar{K}^0 \to \pi^+ e^- \bar{\nu})} = 2.39\times10^{-2},$$

$$\frac{\Gamma(\bar{K}^0 \to \pi^+ \mu^- \bar{\nu}\gamma,\ E_\gamma > 30\ \text{MeV})}{\Gamma(\bar{K}^0 \to \pi^+ \mu^- \bar{\nu})} = 2.17\times10^{-3},\ 2.35\times10^{-3}.$$

It should be emphasized that, because of the infrared divergence in the radiative rate, these results are very sensitive to the value of the minimum photon energy which the experimental apparatus can detect, as is evident from Table I. Indeed, for $E_{\min} = 20$ MeV, the branching ratios involving electrons change to 2.82×10^{-2} and 3.29×10^{-2}, respectively.

In Figs. 1 and 2, we show typical photon spectra. Note that the radiative rate is larger for the $\bar{K}^0$ than for the K^- mode. This is qualitatively what we expect, since the rate for the radiation emitted depends on the mass of the radiating particle. For electron modes the radiation from the electron dominates and there is little difference between the two cases, but for the muon modes we see that the $\bar{K}^0$ case (with muon and pion radiating) is a factor of 3 larger than the K^- case (with muon and kaon radiating).

Next we consider the order of magnitude of the terms neglected in the above formulas. The contribution of the structure-dependent terms was discussed in some detail in Ref. 2, where we retained terms in the square of the matrix element coming from the interference of the infrared-divergent term and the dominant structure-dependent terms. Then the hypothesis of partial conservation of axial-vector current and current-algebra techniques were used to give a rough estimate of the magnitude of the coefficients of these terms, with the result that their contribution to the rate is very small, of the order of 1-2%. The terms proportional to $\lambda_\pm^2$ can be estimated by comparing, for example, the f_+^2 term with the $f_+^2\lambda_+$ term coming from the expansion

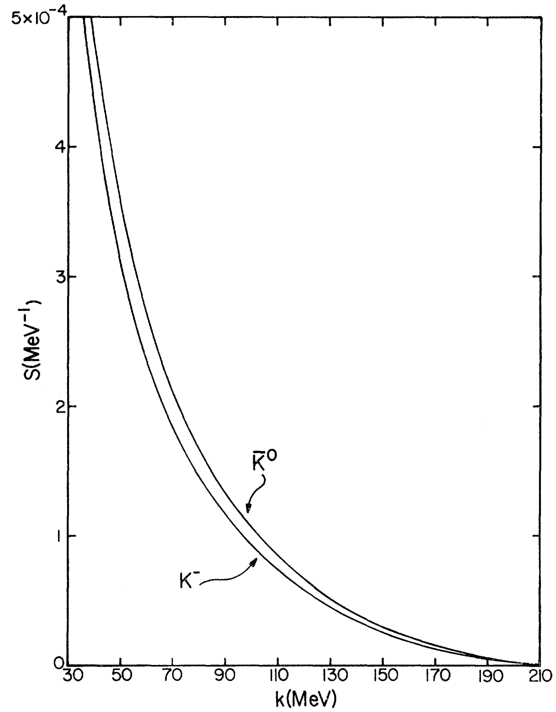

FIG 1. Photon spectrum $S=[d\Gamma(K\rightarrow\pi e\nu\gamma)/dk]/\Gamma(K\rightarrow\pi e\nu)$ for charged ($\lambda_+=0.029$, $\lambda_-=0$) and neutral ($\lambda_+=0.019$, $\lambda_-=0$) kaons.

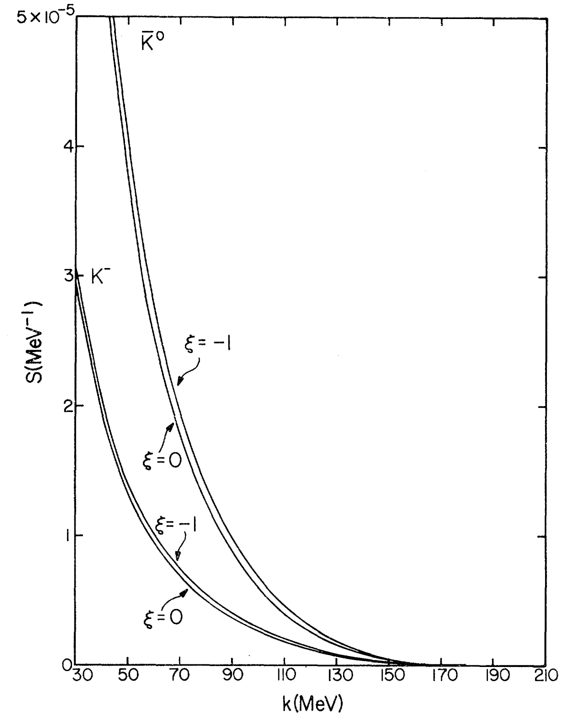

FIG. 2. Photon spectrum $S=[d\Gamma(K\rightarrow\pi\mu\nu\gamma)/dk]/\Gamma(K\rightarrow\pi\mu\nu)$ for charged ($\lambda_+=0.029$, $\lambda_-=0$) and neutral ($\lambda_+=0.019$, $\lambda_-=0$) kaons. The upper curves correspond to $\xi=-1$, and the lower ones to $\xi=0$.

of $f_+(t)$.[2,3] They appear to be of the order of 1-2%, although, in an unfortunate situation where they all add, they could be slightly larger, say 4%. Thus in both cases the corrections to the rates given above would be expected to be a few percent. If larger experimental deviations from the rates given above are seen, it would perhaps indicate the presence of structure-dependent terms larger than we have estimated. In certain regions of the photon spectrum, particularly at the high-energy end, the percentage corrections estimated above could be much larger.

At present, the experimental information is very meager, consisting of one experiment on K^+ decays[10] and one on $K_L{}^0$ decays.[1] The numbers quoted by these groups are

$$\frac{\Gamma(K^+\rightarrow\pi^0e^+\nu\gamma,\ E_\gamma>30\text{ MeV})}{\Gamma(K^+\rightarrow\pi^0e^+\nu)}=(1.2\pm0.8)\times10^{-2}$$

and

$$\frac{\Gamma(K_L{}^0\rightarrow\pi^\pm e^\mp\nu\gamma,\ E_\gamma>?)}{\Gamma(K_L{}^0\rightarrow\pi^\pm e^\mp\nu)}=(0.75\pm0.4)\times10^{-2}.$$

In the latter case the photon cutoff was not specified and the authors state that there could be large systematic effects in their experiment which have not yet been fully analyzed. Although our results, assuming a cutoff at 30 MeV, are higher than these numbers, it is too early to say that any real disagreement exists. It is of importance, however, to clarify this point. If the difference is to be taken at its face value, it suggests that either the structure-dependent terms are much more important than expected[2,3] or that in the experiments some radiative events have been missed entirely or incorrectly classified as nonradiative events. Obviously additional experiments specifically designed to pick out these radiative modes are necessary.

We wish to thank R. W. Brown, E. S. Ginsberg, A. K. Mann, F. A. Nezrick, and H. Primakoff for helpful discussions.

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*Work supported in part by the U. S. Atomic Energy Commission under Contract No. AT(30-1)-3668B.

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[1]G. R. Evans, M. Golden, J. Muir, K. J. Peach, I. A. Budakov, H. W. K. Hopkins, W, Krenz, F. A. Nezrick,

and R. G. Worthington, Phys. Rev. Letters 23, 427 (1969).

[2]E. Fischbach and J. Smith, Phys. Rev. 184, 1645 (1969).

[3]H. W. Fearing, E. Fischbach, and J. Smith, "Current Algebra, $\bar{K}_{l3}{}^0$ Form Factors and Radiative $\bar{K}_{l3}{}^0$ Decay" (to be published).

[4]N. Cabibbo, in Proceedings of the Thirteenth International Conference on High Energy Physics, Berkeley, 1966 (University of California Press, Berkeley, Calif., 1967), pp. 33-34, has given an equivalent set of numbers. We disagree slightly with him on the first number in Eqs. (4) and (5) and cannot check because he does not state which masses were used in the numerical computation.

[5]F. E. Low, Phys. Rev. 110, 974 (1958).

[6]S. L. Adler and Y. Dothan, Phys. Rev. 151, 1267 (1966).

[7]T. H. Burnett and N. M. Kroll, Phys. Rev. Letters 20, 86 (1968).

[8]We checked these results using SCHOONSHIP, a CDC-6600 program for symbolic evaluation of algebraic expressions, written by M. Veltman.

[9]See C. Rubbia, in Proceedings of the Topical Conference on Weak Interactions, CERN, Geneva, Switzerland, 14-17 January 1969 (CERN Scientific Information Service, Geneva, Switzerland, 1969), p. 227.

[10]E. Bellotti and A. Pullia, quoted by W. J. Willis, in Proceedings of the International Conference on Elementary Particles, Heidelberg, Germany, 1967, edited by H. Filthuth (North-Holland Publishing Company, Amsterdam, The Netherlands, 1968), p. 278, and by D. Cline, unpublished. The value 30 MeV for the minimum photon energy was given by Cline.

ERRATA

DETECTION OF NONRESONANT NEUTRON CAPTURE IN Pb^{207} VIA THE THRESHOLD PHOTONEUTRON CROSS SECTION FOR $Pb.^{208}$ C. D. Bowman, R. J. Baglan, and B. L. Berman [Phys. Rev. Letters 23, 796 (1969)].

The direct cross section $\sigma_{\rm dir}$, determined in this work to be 1.2 mb/sr, is the (γ, n) direct cross section $\sigma_{\gamma n}$ and not $\sigma_{n\gamma}$ as implied. $\sigma_{n\gamma} = \sigma_{\gamma n}(k_\gamma{}^2 g_n/k_n{}^2 g_\gamma) = 0.45$ mb/sr.

The definition of angle α was omitted: $\alpha = \tan^{-1}(\mathrm{Im}D/\mathrm{Re}D)$.

On page 799, column 1, line 16, Lane and Lynn's predicted (n, γ) cross section is 4.5 mb/sr, not 0.045 mb/sr. This translates to 0.005 mb/sr at 40 keV as stated in the Letter.

These errata do not change any of the conclusions in the Letter.

OPTICAL DISPERSION AND THE STRUCTURE OF SOLIDS. S. H. Wemple and M. DiDomenico, Jr. [Phys. Rev. Letters 23, 1156 (1969)].

The factor $(2/\pi)^2$ should be omitted from Eq. (6).

MAGNETOMORPHIC SIZE EFFECT IN TUNGSTEN. D. E. Soule and J. C. Abele [Phys. Rev. Letters 23, 1287 (1969)].

On p. 1288, column 1, line 6 should read, "to 1.3°K for this principal electronic band, it is considered⋯."

The last sentence of the caption to Fig. 3 should read, "⋯extremal dHvA orbits."

On p. 1290, column 1, line 22 should read, "the ξ orbit⋯."

On p. 1291, column 1, line 16 of the second paragraph is changed to read "⋯ellipsoids, the effective bandwidth $(\partial R/\partial k_H)^{-2}$, and the amplitude factor R^7 are 0.45, 0.33, and 1.0, respectively, showing the mixed contribution of these limiting point signals."