states in the πN system does not rule out their existence. These states may be weakly coupled to the πN system.

Thus the spin considerations[3] in dual models and the examination of the constraints on the s-channel resonances[12] do provide interesting information concerning the particle spectrum and the coupling patterns. We can easily generalize our results to other systems as well as to particles with arbitrary spin. These considerations and a more detailed discussion of the predictions of local duality will be discussed elsewhere.

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*Work supported by the U. S. Atomic Energy Commission.

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[5]Equation (5) is obtained from Eq. (3) by using the standard symmetry relations for the Clebsch-Gordan coefficients and the d functions.

[6]A particle with arbitrary spin and definite parity has, in general, two couplings to a $(0\frac{3}{2}^+)$ system. The one with the least number of derivatives corresponds to $[\chi_R^{J\pm}(L_0)]^2$. It is assumed that this is the dominant coupling. See, for instance, J. G. Rushbrooke, Phys. Rev. 143, 1345 (1966).

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POSSIBLE FAILURE OF THE POMERANCHUK THEOREM–SHRINKAGE OF THE FORWARD ELASTIC PEAK

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(Received 22 December 1969)

Assuming constant but unequal asymptotic total cross sections $\sigma_\pm(\infty)$ leads to a $(\ln s)^2$ shrinkage of the forward elastic peak. Moreover, under rather general assumptions, $d\sigma^{\rm el}/dt$ exhibits an infinitude of oscillations in t as $s\rightarrow\infty$, a result derived earlier by Finkelstein within the Regge-cut model.

Recent Serpukhov data[1] are consistent with the assumption that constant asymptotic values $\sigma_\pm(\infty)$ of the total $K^\pm N$ cross sections have been achieved at about 30-GeV/c laboratory momentum. If this assumption is correct, it follows that (i) the Pomeranchuk theorem is violated, i.e., $\sigma_-(\infty)\neq\sigma_+(\infty)$, and (ii) the elastic cross sections $d\sigma_\pm^{\rm el}/dt$ increase like $(\ln s)^2$ in the forward direction as $s\rightarrow\infty$. That is, one obtains directly from Pomeranchuk's discussion[2] forward elastic amplitudes of the form

$$A(s,0)=a\ln s+ib\quad(s\rightarrow\infty),\tag{1}$$

where we neglect the effects of spin. {Our normalization is such that $d\sigma^{\rm el}/dt=|A(s,t)|^2$, $b=b_\pm$ is proportional to the total cross section $\sigma_\pm(\infty)$, and $a=a_\pm\propto\pm[\sigma_-(\infty)-\sigma_+(\infty)]$. Both a and b are real constants with $a\neq0$ and $b>0$.} Also, the unitarity condition,

$$\int_{t_1}^0 dt(d\sigma^{\rm el}/dt)\leq\sigma_{\rm tot}^{\rm el}<\sigma_{\rm tot},\tag{2}$$

coupled with the $(\ln s)^2$ increase at $t=0$, implies a $(\ln s)^2$ shrinkage of the forward elastic peak, as has been emphasized independently by several authors recently.[3] Moreover, employing the

Regge-cut model, Finkelstein has obtained an explicit form for $A(s,t)$ with the rather remarkable property that it oscillates wildly with t in the physical region ($t<0$) as $\ln s \to \infty$.[4] His result for $d\sigma/dt$ near $t=0$ oscillates with a frequency proportional to $\ln s$ when plotted versus $(-t)^{1/2}$. We shall show that this infinitude of oscillations in t as $\ln s \to \infty$ follows independently of the hypothesis of Regge cuts and can be obtained from rather general assumptions within the context of the canonical conditions of analyticity, unitarity, crossing, and polynomial boundedness. We are further led to conjecture that these four requirements are sufficient to impose the oscillations in t at fixed s, given asymptotic total cross sections, $\sigma_\pm$, which are constants independent of s but not equal.

We begin with an Ansatz:

$$A(s,t)=(a\ln s+ib)F(x) \quad (|t|<R,\ \ln s\to\infty), \tag{3}$$

where $x\equiv(\ln s)^2(-t)$. It is clear that the unitarity relation (2) is satisfied provided the condition

$$\int_0^{x_1\to\infty} dx\,|F(x)|^2<\sigma_{\rm tot}/a^2 \tag{4}$$

is maintained. Moreover, since $A(s,t)$ is analytic in t in the complex neighborhood of $t=0$, ($|t|<R$),[5] it follows that $F(z)$ is an entire function of z. To reconcile Eqs. (1) and (3), we must have $F(0)=1$. Polynomial boundedness of A,[5] $|A(s,t)|<C|s|^{1-\epsilon}$ as $|s|\to\infty$ for $|t|<R$, imposes a bound on $|F|$ as $|z|\to\infty$. Denoting the maximum value of $|F(z)|$ on the circle $|z|=r$ by $M(r)$, we obtain

$$M(r)<C\exp(\tau r^{1/2})\tau r^{1/2} \quad (r\to\infty), \tag{5}$$

where $\tau=|-t|^{-1/2}$. Finally, the reality condition on $A(s,t)$ below threshold, together with the reflection principle, require $F^*(z^*)=F(z)$.

To utilize some helpful theorems from the theory of entire functions,[6,7] it is useful to consider the order ρ of a function in terms of its maximum circular modulus $M(r)$:

$$\rho\equiv\limsup_{r\to\infty}[\ln\ln M(r)/\ln r]. \tag{6}$$

From Eq. (5) we find for F, $\rho\le\frac{1}{2}$. Next we use an extension of the Phragmén-Lindelöf theorem implying boundedness of a function in a wedge of angle π/ω by a constant, given this condition on the edges and a growth rate $O(\exp r^\rho)$ with $\rho<\omega$.[7] For $\rho<\frac{1}{2}$, the angle can be opened to 2π for $\omega=\frac{1}{2}$, and boundedness along $z=x\ge0$ [implied by Eq. (4)] coupled with Liouville's theorem yields $F(z)$ = const throughout the plane, where $F(z)=1$. But this does not suffice to satisfy Eq. (4) whence we must have $\rho\ge\frac{1}{2}$. Combining the ρ inequalities, $\rho=\frac{1}{2}$. Employing Hadamard's factorization theorem[7] and the condition $F(0)=1$ ($0<\rho<1$), $F(z)$ is determined uniquely by its zeros $\{z_k\}$:

$$F(z)=\prod_{k=1}^{\infty}[1-(z/z_k)]. \tag{7}$$

(The z_k are enumerable and numbered such that $|z_k|$ is a nondecreasing function of k.) Also, since ρ is not an integer, it equals the convergence coefficient ρ_1 of the series $\sum_k|z_k|^{-\alpha}$, where ρ_1 is the infimum of positive α's for which the series converges. It is clear that F has an infinite number of zeros, for if not, α could be zero in the sum, but $\alpha>\rho_1=\frac{1}{2}$. If one assumes that the zeros lie on the positive real axis ($z=x>0$), the infinitude of oscillations in $d\sigma/dt$ discovered by Finkelstein follows immediately. [To obtain closer connection with his work, we note that if $z_k=ck^m$ then, since $\rho_1=\frac{1}{2}$, we must have $m=2$. Substituting in the right side of Eq. (7), and assuming the zeros to be of minimum order (two) consistent with Eq. (4), we obtain[8] for $z=x$

$$F_0(x)=\left(\frac{\sin\pi(x/c)^{1/2}}{\pi(x/c)^{1/2}}\right)^2. \tag{8}$$

Substituting $F_0(x)$ into Eq. (3), we recover Finkelstein's amplitude apart from a normalization factor s and the (usually dominant) Regge factor, $s^t=\exp(-x/\ln s)$, which (here) remains equal to unity over the peak width and many oscillations of $F_0^2(x)$.]

Returning to the general problem, if we drop the assumption that all the zeros lie along the ray $x>0$, our only special assumption is the initial Ansatz, Eq. (3). We shall need essentially one more (see below) in order to prove a series of results. Proceeding in steps, we remove the zeros from the positive real axis, placing them in an initially simple configuration (along the negative real axis), then in increasingly complex ones, showing at each stage that one cannot satisfy the unitarity relation (4). Within the set of configurations considered, an infinitude of zeros must then lie along the physical ray $x\ge0$.

Let $n(r)$ be the number of zeros in and on the circle $|z|=r$. The fact that $\rho=\frac{1}{2}\ne$ integer implies a good deal about the asymptotic behavior of $n(r)$ as $r\to\infty$. To be precise,[6]

$$n(r)=O(r^{\rho+\epsilon}) \quad (\text{all } \epsilon>0 \text{ as } r\to\infty), \tag{9}$$

$$n(r)\ne o(r^{\rho-\epsilon}) \quad (\text{all } \epsilon>0 \text{ as } r\to\infty). \tag{10}$$

Our second assumption is that $n(r)$ increases as some power β of r, $n(r)=\lambda r^{\beta}$ as $r\to\infty$ with $\lambda>0$. The conditions (9) and (10) suffice to show that $\rho-\epsilon\leq\beta\leq\rho+\epsilon$, whence, letting $\epsilon\to 0$,

$$n(r)=\lambda r^{\rho}\quad(r\to\infty;\ \rho=\tfrac{1}{2},\lambda>0). \tag{11}$$

Stage 1.—Assume that all zeros lie along the negative real axis, $z_k=-r_k$. From Eq. (7),[6]

$$\ln F(z)=\sum_{k=1}^{\infty}\ln(1+z/r_k),$$

$$=\int_0^{\infty}dn(\zeta)\ln(1+z/\zeta)=z\int_0^{\infty}\frac{d\zeta\, n(\zeta)}{\zeta(\zeta+z)}. \tag{12}$$

The last equality is obtained by integrating by parts, the integrated part vanishing at the upper limit from Eq. (11), and at the lower since $F(0)\neq 0$, nor is $z=0$ a limit point. One may, with negligible error, substitute the form (11) in Eq. (12) to obtain the asymptotic result at $z=x\to+\infty$,

$$\ln F(x)=\pi\lambda(\csc\pi\rho)x^{\rho}=\pi\lambda x^{1/2}\quad(x\to+\infty). \tag{13}$$

Exponentiating Eq. (13) one sees that the condition (4) is violated. Hence, the zeros cannot be transferred to the unphysical region $x<0$ to avoid the oscillations at positive x without violating unitarity.

Stage 2.—Place the zeros along the ray $\arg z=\theta_1$, where $\pi\geq\theta_1>0$. [For the moment we ignore the requirement, imposed by the condition $F^*(z^*)=F(z)$ and Eq. (7), that there be a conjugate set of zeros along the ray $\arg z=-\theta_1$, i.e., the set $\{z_k\}=\{z_k{}^*\}$]. Then

$$\ln F(z)=\sum_{k=1}^{\infty}\ln(1-ze^{-i\theta_1}/r_k), \tag{14}$$

and $\ln F(z)$ is analytic except for a cut along the ray at θ_1 including the zeros of $F(z)$. Evaluating at $z=x\geq 0$,

$$\ln F(x)=-xe^{-i\theta_1}\int_0^{\infty}\frac{d\zeta\,\lambda\zeta^{\rho}}{\zeta(\zeta-xe^{-i\theta_1})}. \tag{15}$$

Regarded as a function of ζ, the integrand in Eq. (15) is analytic in the ζ plane for any $x>0$ except along the ray at $-\theta_1$ where there exists a cut (also providing for the branch point at $\zeta=0$ induced by the factor ζ^{ρ} for $\rho=\frac{1}{2}$). Therefore, there is no difficulty in converting the integral on ζ from one along the positive real ζ axis to one along the ray $\arg\zeta=\pi-\theta_1$, allowing ready evaluation. The result is

$$\ln F(x)=\pi\lambda(\csc\rho\pi)x^{\rho}\exp[i\rho(\pi-\theta_1)]\quad(x\to\infty). \tag{16}$$

Stage 3.—Place the zeros in conjugate pairs along rays at $\arg z=\pm\theta_1$, thereby satisfying $F^*(z^*)=F(z)$. There are now cuts in the z plane at $\pm\theta_1$. From Eq. (7), at $z=x\geq 0$, we may write $F(x)=F_1(x)F_1{}^*(x)$, where $\ln F_1(x)$ is given by Eq. (16). It is important to note that although there are two cuts in the z plane, in the ζ plane the integrand in Eq. (15) for $\ln F_1(x)$ has only one cut. Therefore the contour deformations performed in evaluating F_1 and $F_1{}^*$ do not interfere with each other. We obtain

$$\ln F_1(x)F_1{}^*(x)=2\pi\lambda_1(\csc\rho\pi)x^{\rho}\times\cos[\rho(\pi-\theta_1)]\quad(x\to\infty). \tag{17}$$

Now, since $\rho=\frac{1}{2}$ and $\pi\geq\theta_1>0$, the argument of the cosine is less than $\frac{1}{2}\pi$, whence

$$\ln F(x)=B_1x^{1/2}\quad(B_1>0;\ x\to\infty). \tag{18}$$

Equation (18) is inconsistent with Eq. (4), forbidding the placement of the zeros along the rays at $\pm\theta_1$ for $\theta_1\neq 0$.

Stage 4.—We extend to $2n$ or $2n+1$ rays at arbitrary angles, $\pm\theta_j$, such that $\pi\geq\theta_j>0$. There are now $2n$ or $2n+1$ cuts in $\ln F(z)$ and it is crucial that no two of these (e.g., the closest to the real positive axis) isolate the physical region $x\geq 0$ from the remainder of the z plane. Passage is allowed through a cut-free region near the origin where there are no zeros [since $F(0)\neq 0$ and the origin is not a limit point]. Also it is not essential that $n_j(r)=\lambda_jr^{\rho}$ on all rays, but only on at least one such ray. It is impossible that the exponent on any ray exceed ρ by Eq. (9). Rays with exponents less than ρ can exist but contribute negligibly to $n(r)$ as $r\to\infty$. Therefore, summing over only rays with $n_j=\lambda_jr^{\rho}$ as $r\to\infty$, we find

$$\ln F(x)=Bx^{1/2}\quad(B>0;\ x\to\infty), \tag{19}$$

where $B=\sum_jB_j$ and $B_j=2\pi\lambda_j\cos\frac{1}{2}(\pi-\theta_j)>0$ since $\pi\geq\theta_j>0$. Again we find inconsistency with unitarity.

Stage 5.—We admit a finite number, N, of zeros along the ray $x>0$. Apply the above arguments to

$$G(z)\equiv F(z)\left[\prod_{k=1}^{N}\left(1-\frac{z}{r_k}\right)\right]^{-1},$$

whence $F(x)$ diverges as $\exp(x^{1/2})$ except in the neighborhoods of the zeros. Again Eq. (4) will be violated.

Stage 6.—Augment the above configurations by allowing an infinite number of zeros along the positive real axis. Of the augmented set, if solutions exist which satisfy Eq. (4), they must contain an infinite number of zeros along $x>0$. Moreover, there exists at least one solution exhibited by Eq. (8).

To summarize, if $\sigma_\pm$ become constant but not equal as $s\to\infty$, the elastic scattering amplitude $A(s,t)$ becomes predominantly real and the differential cross section $d\sigma^{el}/dt$ exhibits a $(\ln s)^2$ shrinkage in t, the momentum transfer. The Ansatz (3) sufficed to show that A has an infinite number of zeros. If all but a finite number of these zeros occur at physical (i.e., real, positive) values of the variable $z=(\ln s)^2(-t)$, then an infinite number of oscillations occur in $d\sigma^{el}/dt$ vs t as t decreases from zero, in accord with the result obtained earlier by Finkelstein within the Regge-cut model. Dropping the assumption of real positive zeros, but assuming a power-law asymptotic behavior of $n(r)$, the number of zeros in $|z|\le r$, we showed that of all possible configurations where the zeros are located on an arbitrarily large but finite set of rays radiating outward from the origin, only those with an infinite number of zeros along $z=x>0$ are consistent with unitarity. Again we recover the infinite number of oscillations in $d\sigma^{el}/dt$ exhibited by Finkelstein's solution. We are led to conjecture that these oscillations are a general feature of a situation where $\sigma_\pm(\infty)$ are unequal if one is to maintain the usual requirements of unitarity, analyticity, crossing, and polynomial boundedness.

I have benefitted from comments by R. J. Eden, J. Finkelstein, D. Horn, D. Narayana, and H. T. Williams. I especially thank R. Kraft and J. A. Shapiro for stimulating discussions.

Note added in proof. – As expected, it is possible to remove the restriction that the zeros of $F(z)$ lie along a finite set of rays. The generalization to an arbitrary distribution of zeros {provided only that the limit of $r^{-\rho}n(r)$ exists as $r\to\infty$ [see Eq. (11)} follows from a lemma due to Levin[9]: Let $\{z_k\}$ be the set of zeros of a canonical product $\Pi(z)$ [of which our $F(z)$ given by Eq. (7) is a special case]. Assume the limit of $r^{-\rho}n(r)$ exists as $r\to\infty$. Let $\Pi^\delta(z)$ denote the same product except for a different set of zeros $\{z_k'\}$ which satisfy $|z_k'|=|z_k|$ and $|\arg z_k'-\arg z_k|<\delta$. Then one can always select δ sufficiently small such that for arbitrary $\epsilon>0$ and $\eta>0$, the inequality

$$|\ln|\Pi(z)|-\ln|\Pi^\delta(z)||<\epsilon r^\rho$$

holds for all z not within an exceptional set of circles C containing the zeros $\{z_k\}$ and $\{z_k'\}$. Moreover, the sum of the radii b_k of all circles in C centered at $|z|<r$ satisfies $r^{-1}\sum_k{}^r b_k<\eta$ as $r\to\infty$. This lemma allows us to replace an $F(z)$ with an arbitrary distribution of zeros by one having its zeros on a finite set of rays with negligible asymptotic error. Thus, for the general case, our proof that there exists an infinite number of oscillations in $d\sigma^{el}/dt$ for constant but unequal $\sigma_\pm(\infty)$ reduces to that already given for the ray configurations of zeros of $F(z)$.

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