MAGNETIC MOMENT OF A PARTICLE WITH ARBITRARY SPIN*

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The magnetic moment of a particle of nonzero spin S is computed by the introduction of minimal coupling into the appropriate Galilean-invariant wave equation. It is shown that the requirement that the differential equations be of first order, together with a minimality assumption on the number of components, uniquely implies a g factor of $1/S$. The role played by this minimality condition is made explicit by means of a counterexample.

One of the significant complications which is encountered in the relativistic treatment of higher spin theories is the rapid increase in the number of components with increasing spin. As a practical consequence of this circumstance one finds that even the calculation of such basic properties as the magnetic moment can be performed only with considerable algebraic difficulty. In the case of the magnetic moment the familiar results for spins $\frac{1}{2}$ and 1 have been extended to $S=\frac{3}{2}$,[1] $S=2$,[2] and $S=N+\frac{1}{2}$,[3] and although the results obtained in all of these cases are consistent with the $1/S$ conjecture of Belinfante[1] for the g factor, there does not yet exist a proof of this hypothesis in the literature. The extreme simplicity of the conjectured form of $g(S)$ suggests, however, that a direct and general attack on the problem should be feasible. In the present note a proof of this result is given for what may be called "minimal" theories which possess the property of Galilean invariance.

It has recently been demonstrated by Levy-Leblond[4] in the spin-$\frac{1}{2}$ case that what was taken to be a triumph of the Dirac theory in predicting the correct g factor for the electron is, in fact, merely a consequence of the requirement that the wave equation be Galilean invariant and of first order in all derivatives. In particular one finds that the assumed form

$$G\psi \equiv \left(Ai\hbar\frac{\partial}{\partial t}+\vec{B}\cdot\frac{\hbar}{i}\nabla+C\right)\psi=0 \tag{1}$$

is Galilean invariant in the spin-$\frac{1}{2}$ case for the following form of the matrices A, $\vec{B}$, and C:

$$A=\tfrac{1}{2}(1+\rho_3),\quad \vec{B}=\rho_1\vec{\sigma},\quad C=m(1-\rho_3),$$

where we use the two commuting sets of Pauli matrices ρ_i and σ_i to span the 4×4-dimensional spinor space. The transformation law for ψ corresponding to the Galilean transformation

$$\vec{x}'=R\vec{x}+\vec{v}t+\vec{a},\quad t'=t+b,$$

is

$$\psi'(\vec{x}',t')=e^{i\hbar^{-1}f(\vec{x},t)}\Delta^{1/2}(\vec{v},R)\psi(\vec{x},t),$$

where we have defined

$$f(\vec{x},t)=\tfrac{1}{2}mv^2t+m\vec{v}\cdot R\vec{x}$$

and

$$\Delta^{1/2}(\vec{v},R)=\begin{bmatrix} D^{1/2}(R) & 0 \\ \frac{1}{2}\vec{\sigma}\cdot\vec{v}D^{1/2}(R) & D^{1/2}(R)\end{bmatrix}$$

with $D^{1/2}(R)$ being the usual two-dimensional representation of spin $\frac{1}{2}$ which acts in the space of the σ matrices. The form of $\Delta^{1/2}(\vec{v},R)$ illustrates the crucial point that the upper components of ψ do not mix with the lower components under a Galilean transformation.

Upon writing ψ in terms of the two-component spinors ϕ and χ,

$$\psi=\begin{pmatrix}\phi\\ \chi\end{pmatrix},$$

one can rewrite Eq. (1) as

$$E\phi+\vec{\sigma}\cdot\vec{p}\chi=0,\quad \vec{\sigma}\cdot\vec{p}\phi+2m\chi=0,$$

and thereby obtain for the case of minimal coupling to an electromagnetic field the equation

$$\left[E-e\varphi-\frac{(\vec{p}-e\vec{A})^2}{2m}+\frac{e\hbar}{2m}\vec{\sigma}\cdot\vec{H}\right]\phi=0$$

which clearly displays the familiar result for the g factor. Corresponding results have been obtained by one of us[5] for spin 1 using techniques familiar in the case of special relativity. The generalization of this approach will be seen to lead to the asserted result.

One proceeds by writing the spin-S wave function as a completely symmetrized $2S$-rank spinor $\psi_{a_1\cdots a_{2S}}$ where each a_i ranges from 1 to 4. In the absence of further restrictions such an object has $\frac{1}{6}(2S+3)(2S+2)(2S+1)$ independent components. The Galilean-invariant Lagrangian can be written as

$$\mathcal{L}=\frac{1}{2S}\int d^3x dt\,\psi_{a_1\cdots a_{2S}}{}^*\left[\sum_{i=1}^{2S}\Gamma_{a_1a_1'}\cdots\Gamma_{a_{i-1}a_{i-1}'}G_{a_ia_i'}\Gamma_{a_{i+1}a_{i+1}'}\cdots\Gamma_{a_{2S}a_{2S}'}\right]\psi_{a_1'\cdots a_{2S}'},\tag{2}$$

where $\Gamma=\frac{1}{2}(1+\rho_3)$ is clearly an invariant matrix since (as has already been observed) upper components do not mix with lower components under Galilean transformations. It is important to note that upon going over to the case of special relativity one replaces Γ by β, and G by $D=\beta(\vec{\gamma}\cdot\vec{p}+m)$.[6] The relativistic theory thus described is therefore guaranteed to have the same magnetic moment as the Galilean-invariant theory being considered.

The wave equation implied by (2),

$$\sum_{i=1}^{2S}\Gamma_{a_1a_1'}\cdots\Gamma_{a_{i-1}a_{i-1}'}G_{a_ia_i'}\Gamma_{a_{i+1}a_{i+1}'}\cdots\Gamma_{a_{2S}a_{2S}'}\psi_{a_1'\cdots a_{2S}'}=0,\tag{3}$$

can be shown[5] to yield

$$\Gamma_{a_1a_1'}\cdots\Gamma_{a_{i-1}a_{i-1}'}G_{a_ia_i'}\Gamma_{a_{i+1}a_{i+1}'}\cdots\Gamma_{a_{2S}a_{2S}'}\psi_{a_1'\cdots a_{2S}'}=0,\tag{4}$$

from which it follows that the number of components in (3) is precisely the same as in the Bargmann-Wigner equations (4). This latter set makes obvious the fact that, because of the occurrence of the $2S-1$ matrices Γ, those components of ψ in which more than one index is a lower index (3 or 4) drop out of the equations. Thus the only Galilean components are those $2S+1$ components in which all indices of ψ are 1 or 2 plus the $4S$ components in which $2S-1$ indices are 1 or 2 and one index takes the value 3 or 4. Using the notation

$$\psi_{a_1\cdots a_{2S}}=\phi_{a_1\cdots a_{2S}}\ \text{for}\ a_i=1,2,$$
$$\psi_{a_1\cdots a_{2S-1}r}=\chi_{a_1\cdots a_{2S-1}}{}^{r-2}\ \text{for}\ a_i=1,2,\ r=3,4,$$

one can now rewrite (2) as

$$\mathcal{L}=\int d^3xdt\left\{\phi_{a_1\cdots a_{2S}}{}^*E\phi_{a_1\cdots a_{2S}}+\frac{1}{2S}\left[\phi_{a_1\cdots a_{2S}}{}^*\sum_{i=1}^{2S}\vec{\sigma}_{a_1r}\cdot\vec{p}\chi_{a_1\cdots a_{i-1}a_{i+1}\cdots a_{2S}}{}^r\right.\right.$$
$$\left.\left.+\sum_{i=1}^{2S}\chi_{a_i\cdots a_{i-1}a_{i+1}\cdots a_{2S}}{}^{*r}\vec{\sigma}_{ra_1}\cdot\vec{p}\phi_{a_1\cdots a_{2S}}\right]+2m\chi_{a_1\cdots a_{2S-1}}{}^{*r}\chi_{a_1\cdots a_{2S-1}}{}^r\right\},$$

while the equations of motion take the form

$$E\phi_{a_1\cdots a_{2S}}+\frac{1}{2S}\sum_{i=1}^{2S}\vec{\sigma}_{a_ir}\cdot\vec{p}\chi_{a_1\cdots a_{i-1}a_{i+1}\cdots a_{2S}}{}^r=0,\quad \vec{\sigma}_{ra_{2S}}\cdot\vec{p}\phi_{a_1\cdots a_{2S}}+2m\chi_{a_1\cdots a_{2S-1}}{}^r=0.\tag{5}$$

The introduction of minimal coupling into the set (5) yields, for the independent components ϕ, the equation

$$(E-e\varphi)\phi_{a_1\cdots a_{2S}}-\frac{1}{4mS}\sum_{i=1}^{2S}(\vec{\sigma}\cdot\vec{\Pi}\vec{\sigma}\cdot\vec{\Pi})_{a_1a_i'}\phi_{a_1\cdots a_{i-1}a_i'a_{i+1}\cdots a_{2S}}=0$$

or

$$(E-e\varphi)\phi_{a_1\cdots a_{2S}}-\frac{\vec{\Pi}^2}{2m}\phi_{a_1\cdots a_{2S}}+\frac{e\hbar}{2mS}\sum_{i=1}^{2S}\left(\frac{\vec{\sigma}}{2}\cdot\vec{H}\right)_{a_ia_i'}\phi_{a_1\cdots a_{i-1}a_i'a_{i+1}\cdots a_{2S}}=0,$$

where we have defined $\vec{\Pi}=\vec{p}-e\vec{A}$. The identification of the g factor is now immediate upon recognition of the fact that under a rotation $\delta\vec{\omega}$, ϕ transforms as a two-component spinor in each index, i.e.,

$$\phi_{a_1\cdots a_{2S}}{}'=\phi_{a_1\cdots a_{2S}}+i\sum_{i=1}^{2S}(\tfrac{1}{2}\vec{\sigma}\cdot\delta\vec{\omega})_{a_ia_i'}\phi_{a_1\cdots a_{i-1}a_i'a_{i+1}\cdots a_{2S}},$$

which form can alternatively be written as

$$\phi_{a_1\cdots a_{2S}}{}'=\phi_{a_1\cdots a_{2S}}+i\,\delta\vec{\omega}\cdot\vec{S}_{a_1\cdots a_{2S};a_1'\cdots a_{2S}'}\phi_{a_1'\cdots a_{2S}'}$$

by definition of the total spin matrices $\vec{S}_{a_1\cdots a_{2S};a_1'\cdots a_{2S}'}$. Thus one has

$$(E-e\varphi)\phi-\frac{\vec{\Pi}^2}{2m}\phi+\frac{e\hbar}{2mS}\vec{S}\cdot\vec{H}\phi=0$$

and the asserted result

$$g(S)=1/S.$$

Inasmuch as the above derivation uses only the minimal $6S+1$ components appropriate to the description of a particle of spin S by a set of first-order differential equations, it is a matter of considerable interest to display the precise role played by this assumption. It will be entirely sufficient to describe only one possible way in which one can increase the number of components since it is found that $g(S)$ can be made entirely arbitrary in the approach to be presented. To this end one introduces a $(2S+2)$-rank spinor $\psi_{a_1\cdots a_{2S}}{}^{r_1r_2}$ which is totally symmetric in the lower $2S$ indices and antisymmetric in its two upper indices. Then the Lagrangian has the form

$$\mathcal{L}=\int d^3x dt\,\psi_{a_1\cdots a_{2S}}{}^{r_1r_2*}\Big\{\tfrac{1}{2}(1-\lambda)[\Gamma_{a_1a_1'}\cdots\Gamma_{a_{2S}a_{2S}'}(\Gamma_{r_1r_1'}G_{r_2r_2'}+G_{r_1r_1'}\Gamma_{r_2r_2'})]$$
$$+\frac{\lambda}{2S}\Gamma_{r_1r_1'}\Gamma_{r_2r_2'}\sum_{i=1}^{2S}\Gamma_{a_1a_1'}\cdots\Gamma_{a_{i-1}a_{i-1}'}G_{a_ia_i'}\Gamma_{a_{i+1}a_{i+1}'}\cdots\Gamma_{a_{2S}a_{2S}'}\Big\}\psi_{a_1'\cdots a_{2S}'}{}^{r_1'r_2},$$

where λ is an arbitrary parameter. Suppressing for the moment all lower indices on ψ, one can write

$$\psi^{r_1r_2}=2^{-1/2}\{\tfrac{1}{2}(1+\rho_3)\phi\sigma_2+i\rho_2\psi^k\sigma^k\sigma_2+\rho_1\psi'\sigma_2\}_{r_1r_2},\tag{6}$$

where ϕ,ψ^k, and ψ' are symmetric multispinors of rank $2S$. In writing Eq. (6) a term proportional to $(1-\rho_3)\sigma_2$ has been omitted since, by the usual argument, terms in which both r_1 and r_2 take the values 3 or 4 do not contribute.

The part of $\mathcal{L}$ proportional to $1-\lambda$ can be written as

$$-\tfrac{1}{2}(1-\lambda)\mathrm{Tr}\psi^*[G\psi\Gamma+\Gamma\psi G^T],\tag{7}$$

with only the components for which $a_i=1$ or 2 in the $2S$ symmetrized indices of φ, ψ^k, and ψ' being relevant because of the $2S$ factors of $\Gamma_{a_ia_i'}$. There are, therefore, apparently $5(2S+1)$ components which survive in the trace (7). Upon calculation of this quantity one finds that it assumes the form

$$\phi^*E\phi-\phi^*p_k\psi^k-\psi^{k*}p_k\phi+2m\psi_k{}^*\psi_k+2m\psi^{*\prime}\psi'.$$

The part of $\mathcal{L}$ proportional to λ is similarly calculated but in this case only the φ part of (6) contributes. In analogy with the previous discussion one now calls ϕ only that part in which all $a_i=1$ or 2, while those components of ϕ for which one component is 3 or 4 are denoted by $\chi_{a_1\cdots a_{2S-1}}{}^r$, where $r=a_{2S}-2$. The total Lagrangian is thus

$$\mathcal{L}=\int d^3x\,dt\Big\{\phi_{a_1\cdots a_{2S}}{}^*E\phi_{a_1\cdots a_{2S}}+(1-\lambda)[\phi_{a_1\cdots a_{2S}}{}^*p_k\psi_{a_1\cdots a_{2S}}{}^k+\psi_{a_1\cdots a_{2S}}{}^{k*}p_k\phi_{a_1\cdots a_{2S}}$$
$$+2m\psi_{a_1\cdots a_{2S}}{}^{k*}\psi_{a_1\cdots a_{2S}}{}^k]+\frac{\lambda}{2S}\Big[\sum_{i=1}^{2S}(\phi_{a_1\cdots a_{2S}}{}^*\vec{\sigma}_{a_ir}\cdot\vec{p}\chi_{a_1\cdots a_{i-1}a_{i+1}\cdots a_{2S}}{}^r$$
$$+\chi_{a_1\cdots a_{i-1}a_{i+1}\cdots a_{2S}}{}^{r*}\vec{\sigma}_{ra_i}\cdot\vec{p}\phi_{a_1\cdots a_{2S}})+2m\chi_{a_1\cdots a_{2S-1}}{}^{r*}\chi_{a_1\cdots a_{2S-1}}{}^r\Big]\Big\},\tag{8}$$

where all spinor summations are now over two-valued indices. The term proportional to $\psi'^*\psi'$ has been omitted since it clearly vanishes as a consequence of the decoupling of ψ'.

The equations implied by (8) are readily found to be

$$E\phi_{a_1\cdots a_{2S}}+(1-\lambda)p_k\psi_{a_1\cdots a_{2S}}{}^k+\frac{\lambda}{2S}\sum_{i=1}^{2S}\vec{\sigma}_{a_i r}\cdot\vec{p}\chi_{a_1\cdots a_{i-1}a_{i+1}\cdots a_{2S}}{}^r=0,$$

$$2m\psi_{a_1\cdots a_{2S}}{}^k+p_k\phi_{a_1\cdots a_{2S}}=0,\quad \vec{\sigma}_{r a_{2S}}\cdot\vec{p}\phi_{a_1\cdots a_{2S}}+2m\chi_{a_1\cdots a_{2S-1}}{}^r=0,$$

which, upon inclusion of minimal coupling, yield as the equation of motion for the independent components ϕ

$$(E-e\varphi)\phi_{a_1\cdots a_{2S}}-\frac{\Pi^2}{2m}\varphi_{a_1\cdots a_{2S}}+\frac{\lambda e\hbar}{2mS}\sum_{i=1}^{2S}\left(\frac{\vec{\sigma}}{2}\cdot\vec{H}\right)_{a_i a_i'}\phi_{a_i\cdots a_{i-1}a_i'a_{i+1}\cdots a_{2S}}=0.$$

This result allows the immediate identification of the g factor as λ/S, thus demonstrating that an increase of the original $6S+1$ variables to $12S+4$ is sufficient to allow one to give a particle of nonzero spin an arbitrary magnetic moment.

In conclusion, it is perhaps useful to note that the replacement of Γ and G by their previously mentioned counterparts in special relativity enables one to write down immediately the corresponding relativistic Lagrangian. The theory thus described in the spin-$\frac{1}{2}$ case has been discussed by Chang[7] in the limit $\lambda=0$. He found that such a Lagrangian describes a particle with no magnetic moment in complete agreement with the results obtained here.

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CURRENT ALGEBRA AND UNITARITY

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A method is presented for constructing current-algebra amplitudes which satisfy (1) threshold theorems, (2) crossing symmetry, (3) approximate unitarity, and (4) cut-plane analyticity, and which reduce to the usual tree approximation in the narrow-resonance limit. Pion-pion scattering is considered in detail to illustrate the method. A derivation of the Kawarabyashi-Suzuki-Riazuddin-Fayyazuddin relation is provided which makes no reference to vector-meson dominance.

It has been known for some time that phenomenological Lagrangians,[1] evaluated in tree approximation, lead to a representation of current algebra which satisfies the appropriate threshold theorems. Difficulties of this approach are that (1) the tree approximation is not unitary, and (2) it is very difficult to compute higher corrections from the nonlinear Lagrangians involved since one is dealing with a nonrenormalizable theory. A treatment of current algebra, parallel to that of the phenomenological Lagrangian, has been developed from the Ward identities of the theory.[2,3] In this paper we show how to continue this program so as to incorporate unitarity into current-algebra models. Pion-pion scattering serves as a useful example to illustrate our view of the subject.

The Ward identities connecting the time-ordered product of four axial-vector currents to $\pi\pi$ scatter-