CLASSICAL DIFFUSION IN A TOKOMAK*

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Analysis of the simplest classical model for resistive diffusion in an axially symmetric Tokomak geometry shows that present experiments are in a transitional regime which requires analysis of the interaction between several distinct, nonlinearly coupled, transient processes. A few consequences of this theory are examined, and a limiting steady state with much simpler scaling is described.

Consider a plasma governed by the classical, macroscopic, resistive magnetohydrodynamic equations with Ohm's law in the form $\vec{E}+\vec{u}\times\vec{B}=\eta\vec{J}$. Although diffusion is inherently time dependent, the standard theory, giving rise to the Pfirsch-Schluter (PS) factor, is a stationary one. It has been pointed out[1] that plasma diffusion is a complex phenomenon with at least two distinct diffusive time scales nonlinearly coupling field diffusion (the skin effect), plasma diffusion, plasma convection, and geometrical effects which are sufficiently subtle to prevent the limit of a long thin torus from approximating a straight system, and to prevent entirely the analysis of any asymmetric system in terms of diffusive processes. The limitation in studying classical diffusion with a steady-state model applies equally well to more sophisticated theories such as banana or "neoclassical" diffusion,[2] but we do not consider these more complex physical models here.

The applicability of this classical theory is enhanced immeasurably by the fact that a number of theoretical predictions turn out to be independent of the value of the resistivity and even, to some extent, independent of whether the dissipative mechanism is classical or anomalous. In particular, granted enough time, the plasma will approach a unique limiting profile (and unique β) which is independent of the value of the resistivity. We find that an appropriate time-dependent generalization of the PS theory is valid for a low-β, axially symmetric Tokomak, but only for large aspect ratio and on a long time scale which is barely reached in present experiments. The alternative Tokomak geometry which we propose should bring the slow time scale and the ultimate steady diffusive equilibrium within a reasonable experimental duration.

We present here a very brief summary of this diffusion theory; the details will be given elsewhere.[3] We restrict our attention to axial symmetry as is appropriate for a Tokomak; moreover, it is easily shown (exactly as in the nondissipative case[4]) that the equations do not admit any solutions at all except in highly symmetric geometries. Introducing

$$\vec{B}=\hat{B}+\vec{B}_\theta,\quad \hat{B}=\frac{1}{r}\vec{e}\times\nabla\psi,\quad \vec{B}_\theta=\frac{\vec{e}f}{r},\tag{1}$$

we have (for an isothermal plasma)

$$\frac{\partial p}{\partial t}+\operatorname{div}(p\vec{u})=0,\tag{2}$$

$$\frac{\partial\psi}{\partial t}+\vec{u}\cdot\nabla\psi=D_0\Delta^*\psi-c,\quad D_0=\frac{\eta}{\mu_0},\tag{3}$$

$$\frac{\partial f}{\partial t}+r^2\operatorname{div}\left(\frac{f\vec{u}}{r^2}\right)-r^2\operatorname{div}\left(\frac{\hat{B}u_\theta}{r}\right)=r^2\operatorname{div}\left(\frac{D_0\nabla f}{r^2}\right),\tag{4}$$

$$\Delta^*\psi=-r^2\mu_0p'-ff',\tag{5}$$

where $\Delta^*=\partial^2/\partial z^2+\partial^2/\partial r^2-(1/r)\partial/\partial r$. Both $p(\psi,t)$ and $rB_\theta=f(\psi,t)$ are, at each instant, constant on ψ contours; $c(t)$ is related to the applied toroidal electric field $\oint\vec{E}\cdot d\vec{x}$.

No direct numerical solution of the system (2)-(5) can be made without substantial preliminary theoretical analysis since the velocity $\vec{u}$ is determined, not by its time derivative, but by the constraint that p and $\vec{B}$ constantly satisfy $\nabla p=\vec{J}\times\vec{B}$. The primary theoretical problem is to find whether such a velocity field is indeed determinate.[3] Heuristically, ignoring convection and setting $\vec{u}=0$, we would observe simple diffusion of the field, $\partial\psi/\partial t=D_0\Delta^*\psi-c$, through a stationary plasma, $\partial p/\partial t=0$. This diffusion coefficient, $\eta/\mu_0=D_0$, corresponds to the ordinary skin effect in a solid conductor of resistivity η. However, the diffusing field and stationary pressure violate the pressure balance, $\nabla p=\vec{J}\times\vec{B}$; in other words, convection cannot be neglected in calculating the skin effect. Nevertheless, we do find a consistent modification of this ad hoc procedure with negligible net plasma motion on this fast time scale. As $\vec{B}$ diffuses through the plasma, p also varies, but in such a way as to keep the volume of a given p contour essentially unaltered.

Another heuristic procedure is to keep the velocity terms but drop all time derivatives (this is equivalent to the Kruskal-Kulsrud postulate[5]). Exploiting this quasistatic hypothesis gives the Maschke modification[6] of the PS value[7] for the diffusive flow, $\oint u_n dS$. We introduce a time-dependent extension of the Kruskal-Kulsrud-Maschke-Pfirsch-Schluter theory by inserting this quasistatic velocity into the mass conservation equation, Eq. (2). This yields an unconventional, inherently nonlinear diffusion equation for p at the slower, "classical" rate (diffusion coefficient $D_1 \sim \beta D_0$). Since time variation of p implies the same for $\vec{B}$, it is not clear that this procedure (keeping $\partial p/\partial t$ and dropping $\partial \vec{B}/\partial t$) is consistent even at arbitrarily low β (which is the conventional justification for the PS recipe). But, as in the previous case involving fast diffusion, a consistent analysis on the slow time scale can be made which in some cases verifies this generalized PS procedure.

These equations show no evident simplification at low β. The reason is that, in the Tokomak geometry, it is the plasma currents which determine the flux surfaces even at arbitrarily low β (we can also expect certain other high-β features in this geometry, such as relative insensitivity to field imperfections, etc.). Deeper analysis shows a simplification in that two distinct time scales appear when we combine large aspect with low β. The most appropriate low-β scaling for a Tokomak is to take toroidal and poloidal components of $\vec{J}\times\vec{B}$ of equal order. Referred to $B_\theta \sim 1$, we take $\hat{B}\sim\epsilon$, $J_\theta \sim \epsilon$, $\hat{J}\sim\epsilon^2$, $p\sim\epsilon^2$. We also take the aspect ratio $A = R/a \sim 1/\epsilon$ (cf. Fig. 1) and conclude that the poloidal energy ratio $\hat{\beta} = 2\mu_0 p/\hat{B}^2$ and the rotation number $\alpha = 1/q \sim R\hat{B}/aB_\theta$ are both unscaled in ϵ. In summary,

$$a, B_\theta, \psi, c, \hat{\beta}, q \sim 1,$$
$$1/R, 1/f, f-f_0, \hat{B}, J_\theta \sim \epsilon,$$
$$p, \beta, \hat{J}, \vec{J}\times\vec{B} \sim \epsilon^2. \qquad (6)$$

The two time scales are $\tau_0 \sim a^2/D_0$ and $\tau_1 \sim \tau_0/\epsilon^2$.

The plasma exhibits a very high degree of diffusional stability. The nondissipative theory allows two functions $p(\psi)$ and $f(\psi)$ to be specified arbitrarily.[8] These functions vary on the fast scale, at the end of which a more restricted quasiequilibrium is set up in which only one function is left free ($R^2p' + ff'/\mu_0 + c/\eta \sim 0$). The single function then varies on the slow scale, ultimately approaching a unique profile (given below). During the fast stage, ψ diffuses through an approximately stationary p profile; then the roles are reversed, and p diffuses slowly through an approximately stationary ψ profile. To verify this stable evolution in time is not trivial, and it is not even always true.[3]

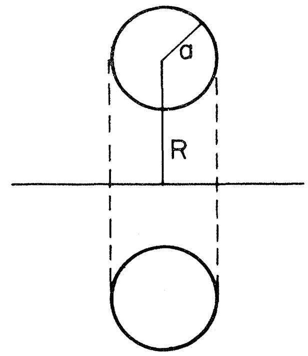

FIG. 1. Tokomak.

That there exists a Pfirsch-Schluter profile with $\oint u_n dS = 0$ has been noticed before.[9] But this special profile is in no way distinguished from the infinity of other PS profiles until its unusual stability is recognized; given enough time, no other profile can be observed. The unique limiting profile is not necessarily a profile which is magnetohydrodynamically stable. If the instability only results in small-scale turbulence (e.g., localized interchanges), it will not alter the diffusional equilibrium profile but will only increase the speed with which it is approached. The same is true for most microinstabilities.

The limiting profile can be calculated in complete generality, with arbitrary β and aspect ratio, and with arbitrary given resistivity and temperature profiles $\eta(\psi)$ and $T(\psi)$ as well as with a volume source of plasma (produced, in an actual Tokomak, by ionization of the background neutral gas). Taking $\sigma(\psi)$ as the distributed mass source,

$$\mathrm{div}(p\vec{u}) = \mathrm{div}(p\vec{u}/RT) = \sigma, \qquad (7)$$

$$\Sigma(\psi) = \int \sigma dV = \int \sigma V' d\psi, \quad \Sigma_1 = \Sigma/pV', \qquad (8)$$

we find

$$R_0^2 \eta p' = -[\lambda^2 c + (1+\lambda^2)\Sigma_1]/[\lambda^2 + (1+\lambda^2)\tau^2], \qquad (9)$$

$$R_0^2 \mu_0 p'/ff' = [\lambda^2 c + (1+\lambda^2)\Sigma_1]/[\tau^2 c - \Sigma_1], \qquad (10)$$

where

$$\lambda^2(\psi) = \langle \hat{B}^2\rangle R_0^2/f^2, \quad R_0^2 = \langle 1/r^2\rangle^{-1}, \qquad (11)$$

$$\tau^2(\psi) = \langle r^2\rangle\langle 1/r^2\rangle - 1, \qquad (12)$$

$$V'\langle\varphi\rangle = \int \varphi dS/|\nabla\psi|, \quad V' = \int dS/|\nabla\psi|. \qquad (13)$$

For each ψ, the parameter λ is a measure of the magnetic aspect ratio and τ of the geometric aspect ratio.

Evaluating the indicated averages for a large-aspect Tokomak with circular section (Fig. 1), and specializing to $\Sigma = 0$ and $\eta =$ constant, we find

$$\tau^2(\psi)/\lambda^2(\psi) = 2q^2 = \text{constant}, \qquad (14)$$

$$\eta p(\psi) = c(\psi_0 - \psi)/R^2(1+2q^2), \qquad (15)$$

$$\hat{\beta}_l = \int \mu_0 p dV / \int \hat{B}^2 dV = 1/(1+2q^2) \qquad (16)$$

(this unusual normalization for $\hat{\beta}$ corresponds to that used in the experimental literature).

The limiting value $\hat{\beta}_l$ is quite small since, in practice, $q \sim 2.5$ or larger. We can obtain much larger values of β_l with an elliptic section (Fig. 2) in which $a/R = O(\epsilon)$ but $b/R = O(1)$. In this case, we find

$$q = \pi B_\theta (a^2+b^2)/\mu_0 IR = \text{constant}, \qquad (17)$$

$$p = I(\psi_0 - \psi)\hat{\beta}_l/\pi abR, \qquad (18)$$

$$\hat{\beta}_l = [1 + 4q^2a^2/(a^2+b^2)]^{-1} \sim 1, \qquad (19)$$

$$\beta_l = [\int 2\mu_0 p dV]/[\int B_\theta{}^2 dV] = b^2\hat{\beta}_l/2q^2R^2, \qquad (20)$$

where I is the total toroidal current.

The limiting profiles are independent of the value of the resistivity and are therefore relatively reliable predictions. Enhanced diffusion caused by large orbit excursions (bananas) will correspond to increased resistivity. Nonthermal small-scale fluctuations will also affect diffusion and resistivity comparably. However, cellular convection or highly ordered fluctuations can decouple diffusion from resistivity.

The experimental data are incomplete and are highly processed with ad hoc theoretical models. Still, there are several conclusions that can be drawn. The first is that in no present experiment is there enough time to reach equilibrium. In particular, the observed value of β is an order of magnitude higher than β_l (even correcting for the influx of ionized neutrals). The expected nonlinear decay ($n \sim 1/t$ rather than e^{-t}) implied by $\beta \gg \beta_l$ has been observed.[10] Particle diffusion

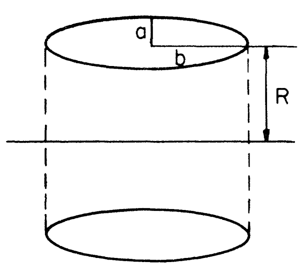

FIG. 2. Flatmak.

(measured only in the TM-3 Tokomak) exceeds by more than an order of magnitude what would be expected at the observed transient value of β and measured resistivity. We conclude, tentatively, that the particle losses in the TM-3 Tokomak are not a consequence of dissipation but of an ordered convective mechanism.

Because of the temperature gradient, the resistivity at the edge of the T-3 plasma can be 10-20 times larger than on axis. The higher diffusion rate at the edge can maintain itself as a transient for the duration of the experiment. Taking this factor, as well as an "anomalous" observed value, 5-10 times high, for the mean plasma resistivity, and the nonlinear ($\beta \gg \beta_l$) transient profile into account would give a diffusion time of 50 msec. This is not inconsistent with the estimated (not measured) lifetime of 20-30 msec,[10] but convection cells are not ruled out. We also propose the resistivity gradient as an explanation for the observed contraction in plasma radius; this is a transient change in the pressure profile rather than an inward motion of the plasma boundary (which is completely ruled out by the present theory on any time scale).

With an elliptic section, Fig. 2, not only is β_l increased, but the relaxation time is reduced. Taking $a/R = 0.15$ and $q = 2.5$, representative of T-3, and choosing $b = R$ ($b/a = 7$) gives a value $\beta_l \sim 0.05$. This is 500 times the theoretical β_l for a circular section and 50 times the experimental value of β. At the same time the relaxation time is reduced by a factor 30 (assuming that the resistivity would be the same), and the total toroidal current carried would be 25 times larger. The comparison of elliptic and circular shapes is made with the same value of q. This is quite arbitrary since for even the circular section the value of q has no simple relation to stability (the common reference to $q > 1$ as a stability margin, referring to the Kruskal-Shafranov criterion, is fallacious since the Kruskal-Shafranov mode, marginal at $q = 1$ in a straight system, remains stable for much larger q in a large-aspect torus).[11]

The Flatmak (favorable large aspect Tokomak – Fig. 2) bears a superficial resemblance to the doublet[12] (Fig. 3). The two are differently motivated – one to gain a diffusive advantage, the other to gain stability. The elliptic section has very little shear (except what may be introduced by a resistivity profile) and has $V'' \sim 0$. The doublet is designed to have high shear and some sort of well. Very little is known about magnetohydrody-

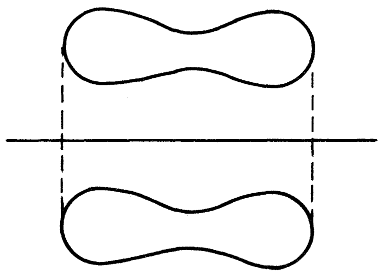

FIG. 3. Doublet.

namic stability for either of these complex shapes; in particular, the relevance of shear or a well is unknown.

To conclude, we repeat that present Tokomak experiments appear to be in a transient state with regard to diffusion. No scaling or extrapolation to new conditions should disregard this fact. The experimentally achieved value of β is a transient (much larger than $\hat{\beta}_l$); any proposed increase in β moves still farther from equilibrium unless the geometry is altered to increase $\hat{\beta}_l$. The limiting profile is, at present, beyond experimental reach, but it could become relevant in an elongated plasma with shorter relaxation time. If so, a significant part of Tokomak scaling would become accessible and relatively insensitive to many parameters.

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PLASMA STABILIZATION BY FEEDBACK

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A simple model is discussed which illustrates the general features of plasma stabilization by an external feedback system. This indicates that different phase relations in the feedback loop are needed to stabilize differing classes of electrostatic instability.

Several experiments have been performed[1-6] in which an otherwise unstable plasma is stabilized by some form of feedback system – that is, by some external electrical system which "senses" the onset of instability and injects an appropriate "suppression" signal.

The nature of the plasma and the way in which the feedback is applied differ considerably in the various experiments and the quantitative realization of feedback stabilization depends on the detailed arrangements. However, a simple model can account for most of the qualitative features of these experiments and also illustrates some significant differences in the application of feedback stabilization to different types of electrostatic instability.

In this model the feedback mechanism is one which senses the potential at r' and in response to this signal charges up a suppressor element at r. The charge on the suppressor is related to the potential of the sensor by a response function

$$g(\omega)G(r, r'),$$

where G is real and $g(\omega)$ is a complex function of the frequency ω which is defined by the amplification $[|g(\omega)|]$ and phase difference $[\arg g(\omega)]$