on experimental conditions, such as the antenna pattern of the probes and the generating frequencies. This dependence on the experimental configuration is due to the fact that the echo is the sum of the electric fields of perturbed free-streaming electrons whose distribution depends in detail upon the interaction between the electrons and the fields of the probes. The ballistic theory of the klystron[2] is currently being used to investigate these effects.

In summary, we have experimentally found longitudinal electron wave echoes whose wavelength-frequency relationship is consistent with a theory which is ballistic in nature. Because of the analytic simplifications afforded by neglecting resonant contributions, this echo may be particularly useful as a diagnostic.

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THERMAL CONDUCTIVITY OF A He^3-He^4 MIXTURE NEAR THE SUPERFLUID TRANSITION

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Measurements are presented which show that the thermal conductivity κ in a He^3-He^4 mixture is finite but singular at the superfluid transition temperature T_λ. From this it can be concluded that dynamic scaling predicts a divergent mass-diffusion coefficient for the normal phase. Through a semiempirical approach, κ for the mixture is shown to be consistent with the divergent κ for pure He^4.

We wish to present high-resolution measurements of the thermal conductivity κ for a solution of 15 mole% He^3 in He^4 ($T_\lambda = 1.946$ K) which show that κ is finite but singular at the superfluid transition temperature T_λ. This work was motivated by the successful detailed explanation[1-3] of the divergence at T_λ of κ for pure He^4 I[4-6] by dynamic scaling. From the absence of a divergence of κ at T_λ and the hydrodynamics applicable to these mixtures[7] it can be concluded that simple dynamic scaling arguments in this case make no prediction[8] about the thermal conductivity, but rather pertain to the divergence of the mass diffusion coefficient for the system in the normal phase. Nonetheless, by means of a reasonable semiempirical expression for κ in mixtures it has been possible to compare the singular contribution to κ with the divergent κ in pure He^4. Using this semiempirical approach, the results for the two systems are found to be consistent in magnitude and temperature dependence.

The measurements were made by the method used previously[6] in the determination of κ for pure He^4, with the exception that the sample length was reduced to 10^{-1} cm. Several power densities were used; but the most extensive results were obtained with a heat flow $Q \cong 6\times10^{-7}$ W/cm^2. Some of the values of κ as a function of the absolute temperature T are shown in Fig. 1. More data over the temperature range indicated in the figure by the horizontal bar near the transition are shown on expanded scales in the insert in Fig. 1.

For $T < T_\lambda$, the measured thermal conductivity can be described by the two-fluid hydrodynamics of mixtures.[9-11] It is an effective thermal conductivity $\kappa_{\rm eff}$ which is a combination of the diffusion coefficient, the thermal diffusion, and the thermal conductivity in the absence of impurity currents. If ideal dilute solution theory applies, $\kappa_{\rm eff} \propto c^{-1}$ (c is the molar concentration of He^3). The ratio between $\kappa_{\rm eff} c$ as measured here for $c = 1.5\times10^{-1}$ and as measured by Ptukha[11,12] at $c = 1.3\times10^{-3}$ varies between 2.5 and 3 over the temperature range covered by the present data

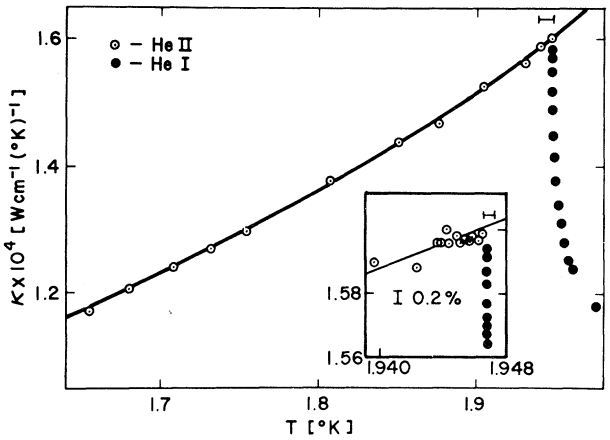

FIG. 1. The measured thermal conductivity κ as a function of the temperature T. The solid line represents a smooth, regular function of T. Measurements over the range indicated by the horizontal bar are shown in the insert. The solid line in the insert is the same regular function of T as that shown in the main figure.

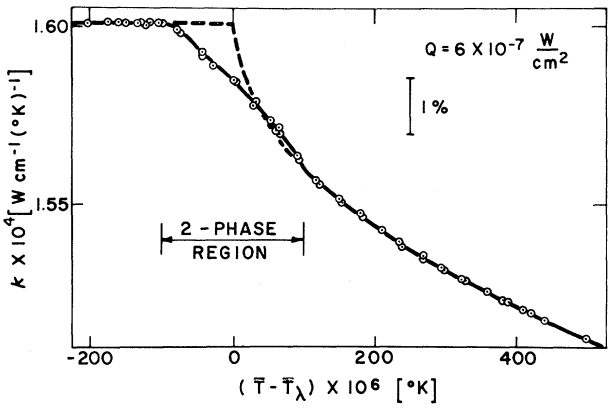

FIG. 2. The measured thermal conductivity κ as a function of $\bar{T}-\bar{T}_\lambda$. $\bar{T}$ and $\bar{T}_\lambda$ are the average sample temperature and the average transition temperature. This figure covers the range indicated by the horizontal bar in the insert of Fig. 1.

below T_λ. The deviation of this ratio from unity may be ascribed primarily to deviations from ideal solution behavior. For $1.5\ \mathrm{K} \lesssim T \leq T_\lambda$, κ_{eff} is expected to be an increasing function of T. Although it is not likely that there is a strongly divergent contribution near T_λ to κ_{eff}, a weak singularity in κ_{eff} as T_λ is approached from $T<T_\lambda$ cannot be ruled out on theoretical grounds.[13] In order to search for a possible singular contribution, a smooth line, representing a regular function of T, was drawn through all the data below T_λ in Fig. 1, and was reproduced in the insert. It is evident from the data that any singular contribution to κ_{eff} for $T<T_\lambda$ is smaller than the experimental error and does not exceed 0.2% of κ_{eff} over the temperature range covered by the measurements ($T_\lambda - T \gtrsim 10^{-4}$ K).

For $T>T_\lambda$, the thermal conductivity decreases with increasing temperature. It is evident from Fig. 1 that κ for He I is singular as T approaches T_λ from higher temperatures. In order to show the temperature dependence of κ near T_λ more clearly, additional data over the temperature range indicated by the horizontal bar in the insert in Fig. 1 are shown on an expanded scale in Fig. 2.

Before proceeding to the comparison of the present results with dynamic scaling and the measurements for pure He^4, it is important to consider the effect of the concentration gradient due to the temperature gradient upon the measurements. For He II, one has for ideal dilute solutions[10,11]

$$\nabla c/\nabla T = -(M_4/RT)\sigma_{40} \cong -0.20\ (\mathrm{K})^{-1}, \qquad (1)$$

where R is the gas constant, σ_{40} is the entropy per gram of pure He^4, and M_4 is the molar mass of He^4. We have estimated the effect of ∇c upon the measured κ for He II, using ideal solution theory, by a method similar to that used by Khalatnikov and Zharkov,[9] and found that for the experimental conditions used here the measured κ differs from that expected in the limit of zero concentration gradient by less than 0.1%.

Very near T_λ, the interpretation of the measurements with nonzero heat current is more complicated. The finite κ results in a nonzero ∇T even for He II, and a He II-He I two-phase system is expected to exist over a relatively large temperature range. In the absence of a concentration gradient this range would just span a temperature interval $T_h - T_c$, where T_h and T_c are the hot and cold end temperatures. However, the existence of a nonzero ∇c results in a transition temperature gradient ∇T_λ, and $\nabla T_\lambda/\nabla c \cong 1.6$ K.[14] Then, for ideal solutions, $\nabla T_\lambda \cong 0.32 \times \nabla T$. The two-phase region should exist over a temperature interval $(T_h - T_c) - (T_{\lambda h} - T_{\lambda c})$. The apparent discontinuities in $d\kappa/dT$ which are evident in Fig. 2 indicate that such a two-phase region did indeed exist over a range of $(2.0 \pm 0.2) \times 10^{-4}$ K at a heat flow Q of 6×10^{-7} W/cm^2. The total measured $T_h - T_c$ under these conditions was 4.12×10^{-4} K. Thus we estimate that for the real solution $\nabla T_\lambda/\nabla T = 0.52 \pm 0.05$, or only somewhat larger than predicted for ideal solutions. When

discussing the singular κ for He I, we shall limit ourselves to the single-phase region, and consider κ as a function of $\overline{T}-\overline{T}_\lambda$, where $\overline{T}$ is the average sample temperature, and $\overline{T}_\lambda$ the average transition temperature. A small curvature correction which compensates for the temperature dependence of κ and the nonzero T_h-T_c (less than 0.2% of κ) was applied to the He I data.

In order to examine the measured thermal conductivity from the viewpoint of dynamic scaling theory,[3,8] we consider the expressions for the normal modes in mixtures, as given by Griffin.[7] These are combinations of mass-diffusion and heat-conduction modes with diffusivities D_A and D_B given by

$$\tfrac{1}{2}\{(1+Z_2)D+D_T\pm[(D-D_T)^2+2Z_2D(D+D_T)+Z_2^2D^2]^{1/2}\}, \tag{2}$$

where the positive sign yields D_A and the negative sign gives D_B. Here $D_T=\kappa\rho^{-1}C_{p,c}^{-1}$ is the thermal diffusivity, ρ the density, $C_{p,c}$ the heat capacity at constant pressure and concentration, and D is the mass-diffusion coefficient.[15] The parameter Z_2 is finite, vanishes for $c=0$, and can be obtained by comparing Eq. (2) with Eq. (15) of Ref. 7. One of D_A and D_B is predicted[3] by dynamic scaling to diverge asymptotically as $u_2\xi$, where the coherence length $\xi\sim\rho_s^{-1}$. Here u_2 is the second-sound velocity, and ρ_s the superfluid density. The other of D_A and D_B is expected to be less singular. Since D_T is known from Ref. 12 and the present experiment to be finite at T_λ provided $c>0$, it follows from Eq. (2) that a divergence of D_A or D_B can be obtained only if D diverges. A divergent D and finite D_T result in a divergent D_A and a finite D_B.[8] Hence, for $c>0$, we expect $D\gg D_T$ sufficiently near T_λ. Therefore one can expand Eq. (2) in terms of D_T/D, and

$$D_A=(1+Z_2)D+Z_2(1+Z_2)^{-1}D_T+Z_2(1+Z_2)^{-3}D_T^2D^{-1}+O(D_T^3D^{-2})+\cdots, \tag{3}$$

$$D_B=(1+Z_2)^{-1}D_T-Z_2(1+Z_2)^{-3}D_T^2D^{-1}+O(D_T^3D^{-2})+\cdots. \tag{4}$$

For $c>0$, it follows that[8]

$$D_A\sim D;\quad D_B\sim D_T. \tag{5}$$

Thus, simple dynamic-scaling arguments make a prediction about the divergence of D, but cannot yield the behavior of D_T, contrary to the statement in Ref. 3.[8]

Since dynamic scaling does not provide a specific prediction with which the present measurements can be compared, we shall use a more empirical approach in order to obtain a reasonable functional form for κ. For $T>T_\lambda$, we can attempt to extrapolate from finite c to the known behavior at zero concentration. Let us write

$$\kappa=a(c)/c[1+g(\epsilon,c)], \tag{6}$$

where $a(c)\equiv\kappa_\lambda c$ (κ_λ is the thermal conductivity at T_λ), and where g is a general function of $\epsilon\equiv(T-T_\lambda)/T_\lambda$ and c. We shall now assume that for sufficiently small c, dilute-solution theory applies for He II even at T_λ, and that for larger concentrations the deviations of κ_λ from the dilute-solution prediction are a regular function of c. In that case, Eq. (6) shows explicitly the expected concentration dependence as c vanishes, and for $c>0$, $a(c)$ is a regular function of c.[16] We now consider the limit of κ for $\epsilon>0$ as c vanishes. In order to obtain a finite κ, one needs at constant $\epsilon>0$

$$g(\epsilon,c)\sim(1/c)f(\epsilon), \tag{7}$$

and for small concentrations

$$\kappa\cong a[c+f(\epsilon)]^{-1};\quad f(\epsilon)>0 \text{ for } \epsilon>0. \tag{8}$$

For sufficiently small c, Eq. (8) must approach the dynamic-scaling prediction[1,3] and experimental measurement[6] for pure He^4, and we expect

$$\lim_{c\to0}[a(c)f^{-1}(\epsilon)]=A\rho C_{p,c}u_2\xi, \tag{9}$$

where A is a numerical constant of order unity. We suggest that even for $c>0$ it is reasonable to compare af^{-1} with $\rho C_{p,c}u_2\xi$. In this case,

$$\kappa_s\equiv af^{-1}=[\kappa^{-1}-\kappa_\lambda^{-1}]^{-1}$$

$$\cong[a(c)/a(0)]A\rho C_{p,c}u_2\xi. \tag{10}$$

At T_λ, κ_s diverges. Experimental values of κ_s for $\epsilon>0$ are shown in Fig. 3. Also shown in Fig. 3 as a solid line are the results for κ of pure He^4,[6] which are known to agree with dynamic scaling.[1,3] In this case, $\kappa_\lambda^{-1}=0$ and $\kappa_s=\kappa$. It is evident that the dependence of κ_s upon ϵ for the mixture is rather similar to that found for κ in pure He^4. At the same value of ϵ, κ_s is larger than κ for He^4 by a factor of about 8. We estimate for ideal solutions,[10,11] and on the assumption that ρ_s/ρ and ξ at the same value of ϵ are independent of c, that

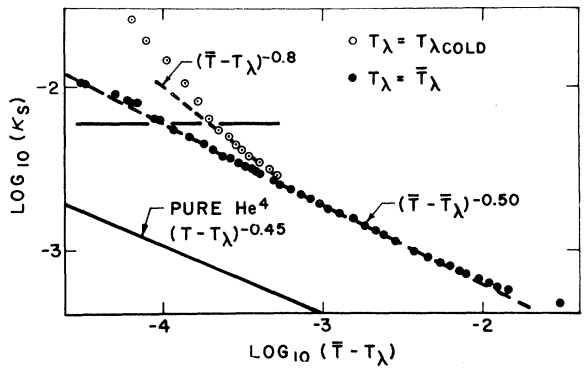

FIG. 3. The singular contribution to κ [defined by Eq. (10)] in W cm^{-1} $(K)^{-1}$ as a function of $\bar{T}-T_\lambda$ in K on logarithmic scales. The data above the horizontal line pertain to the two-phase region. The open circles show the effect of erroneously choosing T_λ as the transition temperature at the cold end of the sample.

$\rho C_{p,c} u_2 \xi$ for $c = 0.15$ is not much different than for pure He^4. In addition, from the experimental data for He II, $a(c)/a(0) \cong 3$ for $c = 0.15$. Thus, the amplitude of the divergence for κ_s differs from the amplitude estimated from the pure He^4 results or from Eq. (10) only by about a factor of 3. We therefore conclude that both the amplitude and temperature dependence of κ_s are reasonably consistent with the He^4 results and Eq. (10) even for $c > 0$. It would be interesting to see if the behavior of Eq. (10) and Fig. 3 could be obtained by a detailed calculation.

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[13]The hydrodynamic expression for κ_{eff} (Ref. 10) contains parameters which could be singular at T_λ.

[14]For numerical data of T_λ vs c, see for example K. W. Takonis and R. de Bruyn Outboter, in *Progress in Low Temperature Physics*, edited by C. J. Gorter (North-Holland, Amsterdam, 1964), Vol. 4, Chap. 2, p. 52.

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[16]The relation $\kappa_\lambda \sim c^{-x}$, $x = 1$, was assumed for the sake of simplicity, and in accordance with the behavior of ideal dilute solutions. If $x \neq 1$, Eq. (10) still follows. The value $x = 1$ is supported by the results reported in Ref. 12 for $T < T_\lambda$, and the absence of a singular contribution near T_λ to $\kappa_{eff}(\epsilon)$ at constant $c = 0.15$ as reported here.