ments is that this contribution, especially to the L transitions, might be somewhat larger than theoretically calculated. This conclusion is supported also by earlier measurements[11] on the same x-ray cascades in Tl as well as by preliminary results of similar measurements in Ta.[12]

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SUPPRESSION OF A PLASMA INSTABILITY BY THE METHOD OF "ASYNCHRONOUS QUENCHING"

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Recent work has shown that an ion-sound instability in a plasma behaves in a manner similar to a classical Van der Pol oscillator. Experimental results are presented which show that the classical method of "asynchronous quenching of a Van der Pol oscillator" can be used to suppress this instability in a plasma. Further, the effects observed as the amplitude and/or frequency are changed are compared with theories due to Minorsky and to Kobsarev (or Bogolyubov).

Recently there have been a number of papers[1-4] which have shown both theoretically and experimentally that various types of self-excited oscillations (or instabilities) in a plasma behave in a manner similar to those described by a classical Van der Pol oscillator.[5] This had led to speculation that the method of "asynchronous quenching of a Van der Pol oscillator"[6-9] might be used to suppress (or quench) an instability present in a plasma. Essentially, the method relies on driving the oscillator (or instability) at a high frequency ω such that $\omega \gg \omega_0$, the oscillator frequency. Then, as the drive amplitude is increased the system behaves as if the asynchronous action of this frequency (ω) were destroying or "quenching" the previously existing self-excited oscillation (ω_0). Minorsky[8] suggested that "asynchronous quenching" could be interpreted as an asymptotic loss of stability of the self-excited oscillation caused by random disturbances due to the imposed high-frequency signal and, further, that as the drive frequency (ω) is increased the amplitude of this oscillation in the system at ω should fall proportionally to $1/\omega$.

In the last few years, there has been some discussion[10-12] on this subject of quenching. It has been suggested[10] that theoretically the phenomena known traditionally as "synchronization"[4,5] and "asynchronous quenching" are the same and, in fact, the "asynchronous quenching" is just "synchronization of a Van der Pol oscillator" at large frequency de-tuning. Further, it has been suggested that the Minorsky stroboscopic analysis[7,8] is incorrect and that when the self-excited oscillation (ω_0) is just suppressed, the amplitude of the signal at frequency ω remaining in the system is constant independent of applied frequency. The results of some analog computer experiments[12] have added weight to this argument. However, this paper reports quenching experiments which have been performed on an ion-sound instability[4] in a neon plasma that, in general, appear to confirm the theory of "asynchronous quenching," but in detail is at variance with the theories of both Minorsky[7,8] and Kobsarev (or Bogolyubov).[6,10]

Theory. – In Ref. 4, it was shown that the density perturbations n_1 of the ion-sound instability in a plasma could be described by a Van der Pol[5] type of equation given by

$$(d^2n_1/dt^2)-(dn_1/dt)(\alpha-2\beta n_1-3\gamma n_1^2)+\omega_0^2 n_1=B\omega^2\sin\omega t, \tag{1}$$

where ω is the applied drive frequency of amplitude B, $\omega_0\,(=k_z c_s)$ is the instability frequency, k_z is the axial wave number, and c_s is the ion-sound speed. Also, $\gamma n_1^2 \ll \beta n_1 \ll \alpha \ll \omega_0$, where α is the linear growth rate and β and γ are the nonlinear saturation coefficients which limit the final amplitude of the unperturbed oscillation A_I $[A_I=(4\alpha/3\gamma)^{1/2}]$.

Then, by making the substitutions $\omega_0 t=\tau$ and $y=2n_1/A_I=(3\gamma/\alpha)^{1/2}n_1$, the reduced Van der Pol equation is obtained:

$$\frac{d^2y}{d\tau^2}-\frac{\alpha}{\omega_0}\left(1-\frac{A_I\beta y}{\alpha}-y^2\right)\frac{dy}{d\tau}-y=E\sin\nu\tau, \tag{2}$$

where $\nu=\omega/\omega_0$ and $E=2B\nu^2/A_I$.

Bogolyubov approach. – (a) If the substitution $y=x+U\sin\nu\tau$ is made in Eq. (2), where $U=E/(1-\nu^2)$, the following equation is found:

$$\frac{d^2x}{d\tau^2}+x\frac{\alpha}{\omega_0}[1-(x+U\sin\nu\tau)^2]\left[\frac{dx}{d\tau}+U\nu\cos\nu\tau\right]=\epsilon f\left(\nu\tau,x,\frac{dx}{d\tau}\right). \tag{3}$$

Equation (3) may be solved by the method of "asymptotic expansion" (see Ref. 9 or 13) for the "nonresonant" case (i.e., when $\omega\neq m\omega_0$, m an integer), in the approximation that $\epsilon=\alpha/\omega_0\ll 1$.

The influence of the perturbing force is expressed by the fact that neither the amplitude of oscillation nor the velocity of phase rotation need remain constant in time, and also harmonics and sum and difference frequency terms appear. Hence, a solution of the form

$$x=a\cos\psi+\epsilon u_1(a,\psi,\nu\tau)+\epsilon^2u_2(a,\psi,\nu\tau)+\cdots \tag{4}$$

is assumed and the possible variations in amplitude and phase are taken into account with the terms

$$da/d\tau=\epsilon A_1(a)+\epsilon^2A_2(a)+\cdots, \tag{5}$$

$$d\psi/d\tau=1+\epsilon B_1(a)+\epsilon^2B_2(a)+\cdots. \tag{6}$$

The problem of solution of Eq. (1) reduces to deriving the functions u_1, u_2, $\cdots A_1$, A_2, $\cdots B_1$, B_2, $\cdots$ such that the assumed solution Eq. (4), with a and ψ substituted by functions of time defined by Eqs. (5) and (6), are a solution of the original Eq. (3).

After much tedious algebra, a solution is obtained correct to second order, that

$$x=a\cos\psi+\epsilon[U\nu(4-U^2-2a^2)/4(1-\nu^2)]\cos\nu t, \tag{7}$$

and a and ψ must satisfy the equations

$$\frac{da}{d\tau}=\epsilon\frac{a}{2}\left(1-\frac{a^2}{4}-\frac{U^2}{2}\right),\ \frac{d\psi}{d\tau}=1-O(\epsilon^2). \tag{8}$$

In the limit of high drive frequencies $\nu=\omega/\omega_0\gg 1$, Eq. (8) reduces to

$$d\psi/d\tau=1-\tfrac{1}{16}\epsilon^2+\epsilon^2U^2(\tfrac{17}{16}-\tfrac{27}{64}U^2). \tag{9}$$

Under steady-state conditions $da/dt = 0$ and thus Eq. (8) becomes

$$A^2 = A_I^2 - 2\nu^4 B^2/(\nu^2-1)^2, \tag{10}$$

where A is the amplitude of instability in the presence of the driving signal, and A_I is the unperturbed amplitude. This shows that as the drive amplitude is increased a point is reached at which $A \to 0$, and then the critical value B_c is given by

$$B_c = 2^{-1/2}[1-(\omega_0/\omega)^2]A_I. \tag{11}$$

The instability frequency should change for high-frequency drives according to Eq. (9), and this is given by

$$\Delta\omega = \alpha^2 B^2[17-27(B^2/A_I^2)^2]/4A_I^2\omega_0, \tag{12}$$

where $\Delta\omega$ is the change in frequency from its value when $B = 0$. The signal remaining in the plasma when the instability just disappears is

$$n_1 x 2^{-1/2} A_I \sin\omega t. \tag{13}$$

Summarizing, this approach maintains that when a high-frequency signal ω is applied to a self-excited oscillation A_I, this oscillation should be "quenched" when the critical drive amplitude B_c reaches a value $B_c \simeq A_I/\sqrt{2}$, and the signal remaining in the system at frequency ω is of magnitude $\approx A_I/\sqrt{2}$.

Minorsky approach.[7,8]—(b) Returning to Eq. (2), Minorsky starts by taking a zero-order solution, for the case $\epsilon = 0$ (and uses the system of units such that $E = 1$):

$$y_0(\tau) = A\sin\tau + B\cos\tau + M\sin\nu\tau, \tag{14}$$

$$u_0(\tau) = d(y_0(\tau))/d\tau = A\cos\tau - B\sin\tau + M\nu\cos\nu\tau, \tag{15}$$

where $M = 1/(1-\nu^2)$.

For first order, he takes $y(\tau) = y_0(\tau) + \epsilon y_1(\tau)$; $u(\tau) = u_0(\tau) + \epsilon u_1(\tau)$. From perturbation theory the relationships below are obtained:

$$y_1(\tau) = \int_0^\tau \sin(\tau - g) f(y_0, (dy_0/d\tau))\,dg; \quad u_1(\tau) = \int_0^\tau \cos(\tau - g) f(y_0, (dy_0/d\tau))\,dg, \tag{16}$$

with

$$f(y_0, (dy_0/d\tau)) = [\alpha - (A_I\beta/\alpha)y_0 - y_0^2]\,dy_0/d\tau.$$

Then, going to the "stroboscopic" equations which essentially "see" y and u only at periodic intervals of $2\pi/\nu$, and defining a stroboscopic "time" scale of $T = 2\pi\epsilon$ in order to observe what is happening to the self-excited oscillation, equations can be obtained for dy/dT and du/dT which in the limit of large ν ($\gg 1$) reduce to

$$dy/dT = -(2\pi^2/\epsilon\nu^2)y + (2\pi/\epsilon\nu^2)u + (2\pi/\epsilon\nu^2), \tag{17}$$

$$du/dT = -(2\pi/\epsilon\nu^2)y - (2\pi^2/\epsilon\nu^3)u - (2\pi^2/\epsilon\nu^4). \tag{18}$$

In the steady-state conditions, $dy/dT = du/dT = 0$, and Minorsky obtains the solution

$$y = 0, \quad u = 1/\omega. \tag{19}$$

As the energy E stored in the oscillation is $E = u^2 + y^2 = 1/\omega^2$, the amplitude of the signal falls as $\sqrt{E} = 1/\omega$, when quenching of the self-excited oscillations at ω_0 occurs.

Experimental.—The experimental arrangement was similar to that described in Ref. 4. The plasma was the positive column of a neon arc discharge with a mercury-pool cathode, and had a peak density $n_0 \sim 3\times10^{11}$ cm^{-3}, a constant electron temperature $T_e \simeq 7.0$ eV, and was contained in an axial magnetic field $B_0 \sim 200$ G. The nature of the instability was determined by using radially movable probes and axially movable photodiodes outside the glass containing vessel. It was found to have predominantly a single frequency, independent of magnetic field, of ~7.5 kHz with an $m = 0$ azimuthal mode number, and an axial wavelength $\lambda_z = 80$ cm. The experimental phase velocity was 6.0×10^5 cm/sec, compared with a theoretical ion-second velocity $C_s = (kT_e/M_I)^{1/2} = 5.8\times10^5$ cm/sec. Consequently, as the oscillation was independent of peak density and axial magnetic field, it was identified as an ion-sound instability.

Density perturbations at varying frequencies ω were applied to the plasma by four different methods; these were as follows:

(1) By direct application of an oscillating potential between cathode and anode of the apparatus, which modulated the axial velocity v_z (and consequently the density) of the plasma.

(2) By connecting an oscillator between the two grid probes spaced apart axially in the plasma, the density perturbations being produced by the same mechanism as in (1) above.

(3) By passing oscillating current through four magnetic coils spaced azimuthally at equal intervals around the discharge column. The plane of each coil was such that an in-phase oscillating magnetic field $\tilde{B}_\theta$ was produced in the plasma, which by virtue of the $[\tilde{B}_\theta \times \tilde{E}_r]$ drift caused an oscillating axial velocity v_z (or density perturbation) in the plasma. (Here $\tilde{E}_r$ is the zero-order radial electric field in the plasma.)

(4) By passing current through a single magnetic coil wound around the glass discharge tube, which produced an oscillating field $\tilde{B}_z$ and had the effect of "squeezing" and "relaxing" the plasma, thus modifying the containment pressure p at this position. Consequently, this produced a density perturbation in the plasma, linearly proportional to the current I in the coil (since $\tilde{p} \propto \tilde{n} \propto B_0\tilde{B}_z \propto \tilde{I}$).

All four methods produced essentially the same results and the details presented here were obtained using method (4). The density perturbations in the system were monitored at different radial and axial positions by using either an ion-biased probe or an axially movable photodiode, and the output signal was displayed on a spectrum analyzer. A typical frequency spectrum of the instability, when the external drive signal is zero, is shown in photograph (a) of Fig. 1. Photograph (b) shows the effect when the ac current in the coil (at 85 kHz) is increased to 6.5 A peak to peak and it is seen that the instability is quenched. Equation (10) in Bogolyubov's theory predicts that the square of the amplitude of the instability A^2 should fall linearly with the square of the drive amplitude B^2 ($\propto I^2$). This is shown plotted in Fig. 1 for three different frequencies; 20, 53, and 85 kHz, and in each case the linear relationship appears to be well satisfied. Further, in this theory, Eq. (12) predicts that the critical amplitude $B_c(\omega)$ [or critical current $I_c(\omega)$] should vary as a function of applied frequency ω, proportional to $[1-(\omega_0/\omega)^2]$. Figure 2(a) shows the reduced critical current $I_c(\omega)/I_c(\infty)$ [where $I_c(\infty)$ is the critical current at high frequencies] plotted as a function of drive frequency.

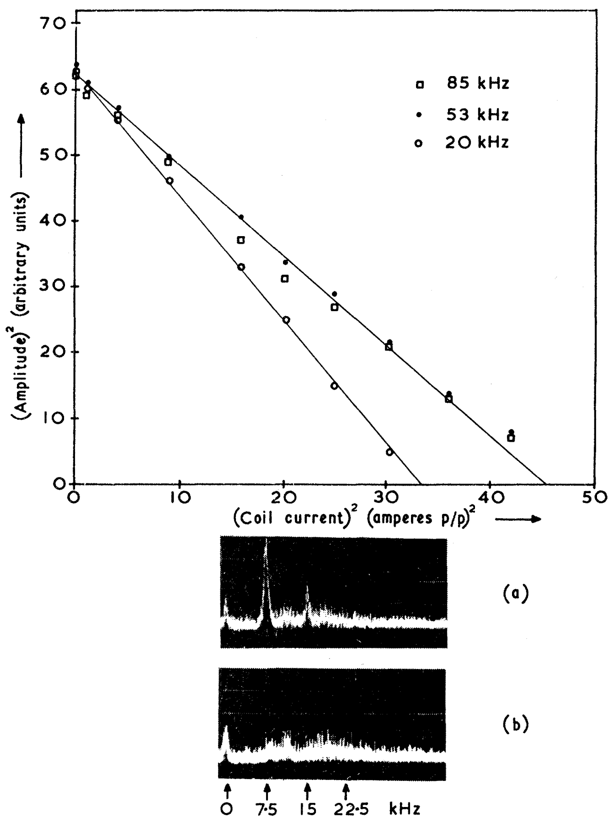

FIG. 1. Experimental amplitude squared (arbitrary units) plotted versus (drive) current squared [A^2 (peak to peak)]. Photographs show frequency spectrum of instability for (a) zero drive current and (b) drive current 6.5 A (peak to peak).

The theoretical prediction Eq. (11) is shown as the continuous lines in this figure. Again, it is seen that good agreement is obtained.

The Bogolyubov theory also predicts that, as the drive amplitude is increased and the instability is gradually quenched, the frequency of the instability should gradually increase from its unperturbed value and that the shift $\Delta\omega$ should be predicted by Eq. (12) for high-frequency drives. The theory shows that this shift $\Delta\omega$ should be proportional to the linear growth rate squared (i.e., α^2). The linear growth rate α was measured by using the method of "asynchronous quenching" to suppress the instability and then utilizing a tone-burst generator to gate the signal in the drive coil at periodic intervals. The resulting instability signal was photographed and the rise time and decay time analyzed to obtain a value for the linear growth rate α. In case (a) it was found to be $(0.20 \pm 0.04)\omega_0$. By modifying the conditions of the arc, the growth rate in case (b) was changed to $(0.33 \pm 0.05)\omega_0$. Figure 3

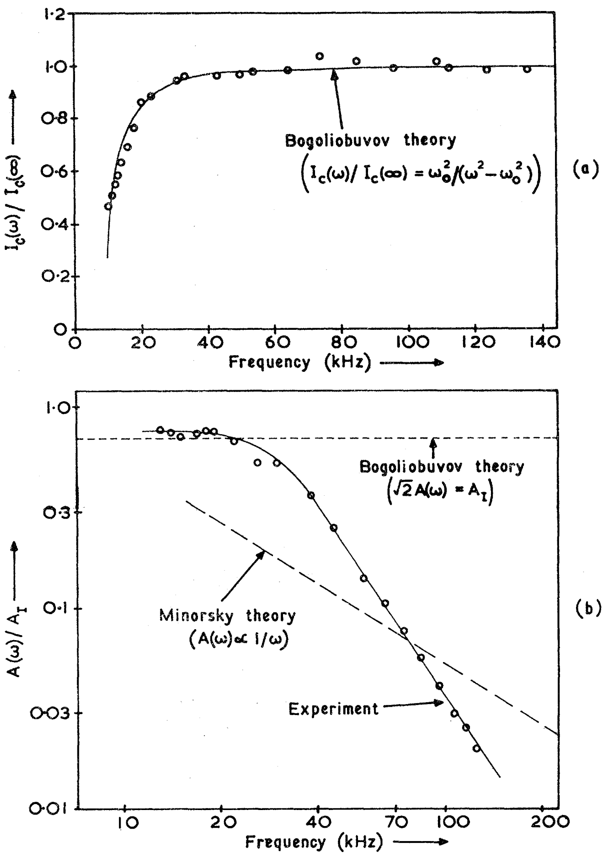

FIG. 2. (a) Reduced critical current $I_c(\omega)/I_c(\infty)$ versus drive frequency. (b) Reduced drive amplitude in the plasma $A(\omega)/A_I$ as a function of drive frequency ω.

shows the shift Δf obtained in the two cases plotted as a function of reduced current squared $(I/I_c)^2$. Again, it is seen that good agreement is found.

Also, the amplitude $A(\omega)$ (at the drive frequency) present in the plasma when the current reaches its critical value $I_c(\omega)$ (and the instability is quenched) has been measured as a function of frequency ω. This is shown in Fig. 2(b) plotted as reduced amplitude $A(\omega)/A_I$ against frequency. Here the agreement with theory ends, and it is seen that the Bogolyubov[9] theory predicts $A(\omega)/A_I \approx 1/\sqrt{2}$ independent of frequency, whereas Minorsky[7] predicts that the amplitude should fall proportional to $1/\omega$ as the frequency is increased. In fact, we find that it falls more rapidly than this and is nearer a $1/\omega^2$ variation.

No evidence was found to support the conjecture of Pengilley and Milner[10] that "asynchronous quenching" and "synchronization" are the same phenomenon. No synchronization, or frequency entrainment, between the drive frequency or its

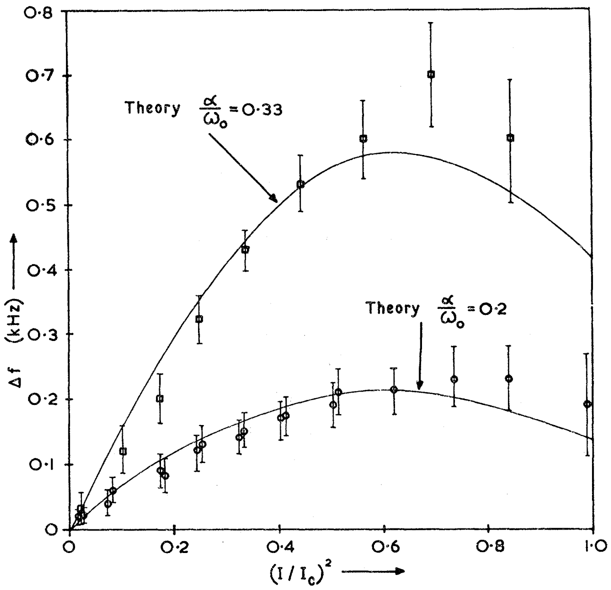

FIG. 3. Frequency change Δf(kHz) as a function of reduced critical current squared $(I/I_c)^2$.

subharmonics and the instability was apparent until the signal frequency approached to within a few kHz of the instability frequency. Generally, the agreement with the Bogolyubov approach is good, and as a consequence it is inferred that "asynchronous quenching" is the mechanism of suppression in these experiments.

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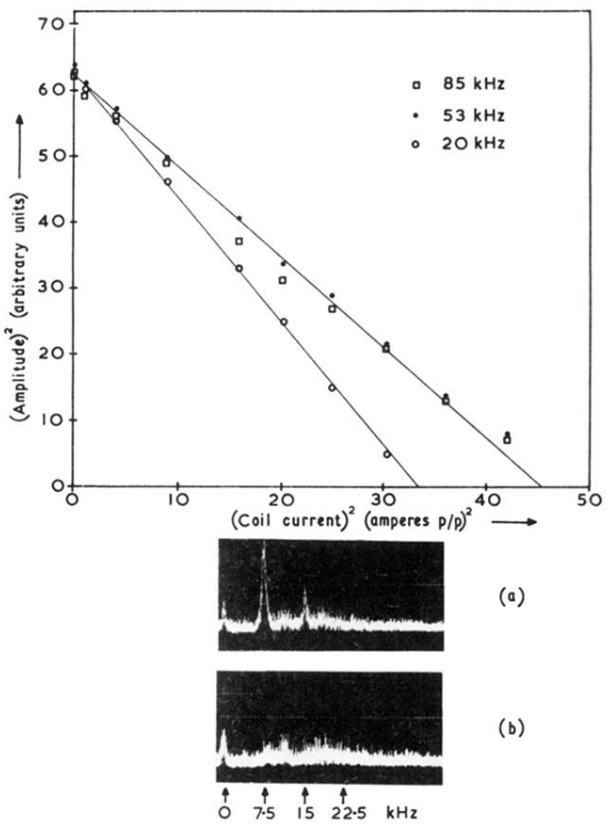

FIG. 1. Experimental amplitude squared (arbitrary units) plotted versus (drive) current squared [A^2 (peak to peak)]. Photographs show frequency spectrum of instability for (a) zero drive current and (b) drive current 6.5 A (peak to peak).