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[10]The peaking of the spectrum at $0.3\omega_c$ may not occur for the real ion-cyclotron wave because of the different dispersion relation, as is mentioned in Ref. 4. However, for the electron-cyclotron wave this effect should exist because of the same dispersion relation as assumed here.

ENERGY SPECTRUM OF A DILUTE HARD-SPHERE BOSE GAS*

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(Received 27 February 1970)

We use linear response theory to calculate the elementary excitation spectrum of a dilute hard-sphere Bose gas. Our result differs by a factor $2^{-1/2}$ from the exact result.

The energy spectrum of elementary excitations corresponding to density fluctuations of a many-particle system is given by the values of energy which correspond to the poles of a generalized susceptibility.[1] Recently Singwi et al.[2] proposed a self-consistent way of calculating the susceptibility of a many-electron system which seemed very promising in treating divergent potentials, but its use was restricted to systems in which the pair distribution function was known in advance.[3] The purpose of this Letter is to apply Singwi's theory to divergent potentials and derive the excitation spectrum from first principles, i.e., without the use of any parameter but the particle-particle potential and the density. We chose the worst possible potential, namely, the hard-sphere interaction, and we take the particles to form a dilute Bose gas.

As in Ref. 2 we take the susceptibility to be of the form

$$\chi(q\omega)=\frac{\chi_0(q,\omega)}{1-\psi(q)\chi_0(q,\omega)}, \tag{1}$$

where χ_0 is the noninteracting susceptibility. In the limit of small q, $\psi(q)$ is given by[4]

$$\psi(q)=-\left(\frac{4\pi}{3q^3}\right)\int_0^\infty drg(r)[\sin(qr)-qr\cos(qr)]\left[\frac{d\varphi}{dr}-\left(\frac{r}{2}\right)\left(\frac{d^2\varphi}{dr^2}\right)\right]. \tag{2}$$

It is very easy to show that, at zero temperature, the relation between the energy of the excitation and $\psi(q)$ is, with $\epsilon(q)=\hbar^2q^2/2m$ and a density n,

$$E(q)=\{[\epsilon(q)]^2+2n\epsilon(q)\psi(q)\}^{1/2}=\epsilon(q)/S(q), \tag{3}$$

the well-known Feynman expression for bosons.[5] To calculate $\psi(q)$ we substitute in Eq. (2) the hard core for φ, putting

$$\frac{d\varphi}{dr}=\lim_{\lambda\to\infty}(-\lambda)\delta(r-r_c),$$

where r_c is the hard-core diam. We get for $\psi(q)$ the following expression:

$$\psi(q)=(2\pi/3q^3)\lambda[g(r_c)\alpha(qr_c)+r_cg'(r_c)\beta(qr_c)], \tag{4}$$

where

$$\alpha(x)=3\sin x-3x\cos x+x^2\sin x,\quad \beta(x)=\sin x-x\cos x.$$

We assume, at this point, that $g(r_c)=g'(r_c)=0$. Thus, the products $\lambda g(r_c)$ and $\lambda g'(r_c)$ are not determined. To find out what their values are we calculate $g(r)$ and $g'(r)$ from $\psi(q)$,[6] and impose that $g(r_c)=g'(r_c)=0$, consistent with our previous assumption.

We are then left with two equations to solve[7] for A and B:

$$1+\frac{2}{3\pi}\left(\frac{r_0}{r_c}\right)^3\int_0^\infty x\sin x\left\{\left[1+\left(\frac{r_c}{r_0}\right)^3\left(\frac{\alpha(x)A+\beta(x)B}{x^5}\right)\right]^{1/2}-1\right\}dx=0, \tag{5}$$

$$1+\frac{2}{3\pi}\left(\frac{r_0}{r_c}\right)^3\int_0^\infty x^2\cos x\left\{\left[1+\left(\frac{r_c}{r_0}\right)^3\left(\frac{\alpha(x)A+\beta(x)B}{x^5}\right)\right]^{1/2}-1\right\}dx=0, \tag{6}$$

where r_0 is the average particle distance and A and B are dimensionless parameters, linear in the unknown $\lambda g(r_c)$ and $g'(r_c)$ and $\lambda g'(r_c)$.

In order to find an analytic solution, we assume $r_c/r_0 \to 0$. Then Eqs. (5) and (6) can be solved exactly and we find $A=12$ and $B=-54$. With these values of A and B we calculate the excitation spectrum or, equivalently, the sound velocity which turns out to be[8]

$$C=(\hbar/\sqrt{2}mr_c)(r_c/r_0)^{3/2}\sqrt{3}. \tag{7}$$

The sound velocity we find in this simple calculation is very close to the exact result,[9] the only difference remaining being the factor $2^{-1/2}$ which does not occur in the exact result.

In conclusion, we show in this Letter that divergences in the potential can be consistently eliminated, and that linear response theory can be used to obtain the excitation spectrum of a hard-sphere Bose gas in a relatively simple way. We specialized for the case of extreme dilution in order to have an analytic solution but, in principle, Eqs. (5) and (6) can be solved for any density.

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*Work supported by the National Research Council of Brazil and Foundation for Research of the State of São Paulo.

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[6]For the relations between $\psi(q)$, $S(q)$, and $g(r)$, see Ref. 2.

[7]We extended the form of $\psi(q)$, valid for low q's, for all q values. The argument is that the low-q part is the most important contribution to Eqs. (5) and (6).

[8]In the small-q region of the spectrum, $E(q)$ is linear with q, as can be easily checked in Eq. (3) since $\psi(q)$ is independent of q. We thus have a sound-wave spectrum where $c=E(q)/\hbar q$.

[9]See Huang, Ref. 5, pp. 409–434.

DECOUPLED-MODE DYNAMICAL SCALING THEORY OF THE BINARY-LIQUID PHASE TRANSITION*

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The critical slowing down of the diffusion in a binary liquid is calculated from the fluctuation-dissipation theorem. The fluctuating current is the product of the local fluctuations in concentration and velocity. Assuming statistical independence of these variables yields results identical to those found by Kawasaki using another method.

The central idea in the dynamical scaling theory[1,2] of phase transitions is that the correlation length is the same for static and dynamic properties. Calculations of the so-called "mode-mode mixing" type[3,4] have been carried out on the binary-liquid phase transition and give a concrete example of how the static correlation length enters the dynamical properties. The purpose of the present note is to point out an alternative approach to the dynamics of the binary-liquid phase transition, which is simply an application of the fluctuation-dissipation theorem to the fluctuations in particle current. By introducing a certain mode-decoupling approximation we obtain