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OBSERVATION OF SELF-TRAPPING INSTABILITY OF A PLASMA CYCLOTRON WAVE IN A COMPUTER EXPERIMENT

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The exact behavior of a modulated cyclotron wave in a plasma is traced in a numerical model to study the self-trapping instability represented by the nonlinear Schrödinger equation with an attractive potential. The instability is found to be explosive in that the wave collapses at a much faster rate than the prediction of the perturbation solution.

In a recent paper,[1] Taniuti and Washimi have shown that a modulated whistler wave becomes unstable owing to its self-action produced by the third-order nonlinearity. They derived a nonlinear Schrödinger equation for the modulation amplitude $\varphi(x,t)$ of a whistler wave represented by $\varphi(x,t)\exp[i(kx-\omega t)]$ and showed that the equivalent potential term is attractive in sign.

If a nonlinear dispersion characteristic $\omega(k,a^2)$ is known, where a is the amplitude of the wave, the modulation amplitude of a finite amplitude wave φ in the nonlinear dispersive medium can generally be expressed by the nonlinear Schrödinger equation of the following form[2]:

$$i\frac{\partial\varphi}{\partial t}+\alpha(|\varphi|^2-|\varphi_0|^2)\varphi+\beta\frac{\partial^2\varphi}{\partial x^2}=0, \qquad (1)$$

where $\alpha=-\partial\omega/\partial a^2$, $\beta=\frac{1}{2}\partial^2\omega/\partial k^2$, and φ_0 is the initial value of φ. The condition of self-trapping can easily be derived from Eq. (1) as the condition of attractive potential,

$$(\partial\omega/\partial a^2)(\partial^2\omega/\partial k^2)<0. \qquad (2)$$

Using this concept Sivasubramanian and Tang[3] have derived the unstable region in the dispersion relation for waves in a cold magnetized plasma. Hasegawa[4] has shown that the sign of the $\partial\omega/\partial a^2$ term, which is crucial in deciding the stability, can be changed arbitrarily in the presence of a coupling to a nondispersive low-frequency mode. Tam[5] and Petviashvile,[6] using somewhat different approaches, have shown, respectively, that obliquely (with respect to magnetic field) propagating whistler and ion-acoustic waves face similar instabilities.

A mathematical solution of Eq. (1) for an attractive potential is not available, however, so the actual time evolution of the instability is still unknown. Because there is a large class of physical problems[7] that are representable by the conspicuous characteristic of the nonlinear Schrödinger equation, it is of great interest to study the dynamic evolution of Eq. (1). In this paper instead of solving Eq. (1), we discuss the results of computer experiments on a dynamic system whose long-time asymptotic behavior is represented by this equation. Thus we can obtain the dynamic behavior not only of the modulation amplitude but also of the wave itself in such a system. The problem we treat is a transverse perturbation of ions propagating parallel to an applied magnetic field in a cold plasma. In the linear regime the perturbation corresponds to the ion-cyclotron wave (this can also be regarded as an electron-cyclotron wave by changing the polarization and the time constant). We use the sheet-current model[8] in which the transverse motion of the charged particles is represented by a set of infinite sheet currents arranged perpendicular to the applied magnetic field.

In this case, the coefficients of Eq. (1) can be shown to be[4]

$$\alpha=-\frac{\omega_c}{8}\left(1+\frac{c^2k^2}{\omega_p^2}\right)^2, \qquad (3)$$

$$\beta=\frac{c^2\omega_c}{\omega_p^2}\frac{1-3c^2k^2/\omega_p^2}{(1+c^2k^2/\omega_p^2)^3}, \qquad (4)$$

for φ representing a wave magnetic field B_1 as

$$\varphi = B_1/B_0, \tag{5}$$

where ω_p and ω_c are ion plasma and cyclotron frequencies, respectively, and B_0 is the applied magnetic field. [For an electron-cyclotron wave use electron-plasma and electron-cyclotron frequencies and multiply Eq. (3) by the electron-to-ion mass ratio.] In deriving Eqs. (3) and (4), electron dynamics are ignored except for their effect in providing charge neutrality to be consistent with the numerical model.[9]

To compare with the numerical result, we first obtain a perturbation solution of Eq. (1) by applying a transformation

$$\varphi = \rho^{1/2} \exp[i \textstyle\int \sigma dx/\beta] \tag{6}$$

and expanding ρ and σ:

$$\begin{pmatrix} \rho \\ \sigma \end{pmatrix} = \begin{pmatrix} \rho_0 \\ \sigma_0 \end{pmatrix} + \begin{pmatrix} \rho_1 \\ \sigma_1 \end{pmatrix} \exp i(Kx - \Omega t). \tag{7}$$

We obtain

$$\rho_0 = \varphi_0^{\,2}, \quad \sigma_0 = 0 \tag{8}$$

and

$$\Omega^2 = \beta^2(K^2 - \alpha\varphi_0^{\,2}/\beta)^2 - \alpha^2\varphi_0^{\,4}. \tag{9}$$

Hence the maximum growth occurs for

$$K \equiv K_m = \varphi_0(\alpha/\beta)^{1/2}$$

and the corresponding growth rate Γ_m is

$$\Gamma_m = \mathrm{Im}\Omega_m = |\alpha|\varphi_0^{\,2}. \tag{10}$$

From Eq. (9) one can see that for a real K, Ω^2 becomes negative only when $\alpha\beta > 0$, which is in agreement with Eq. (2). The growth rate Γ is proportional to the square of the initial wave amplitude φ_0.

The computer experiment is performed with periodic boundary condition with the periodic length L given by

$$L = 2\pi/K_m. \tag{11}$$

The wave number k of the carrier wave is so chosen that the growth rate Γ_m becomes maximum for a given ratio of k/K_m, giving

$$ck/\omega_p = 1.59. \tag{12}$$

Other parameters are $\omega_p/\omega_c = 0.629$, $|\varphi_0|$ $(=|B_1/B_0|$ at $t=0) = 0.121$, $k/K_m = 4$, $(\rho_1/\rho_0)_{t=0} = 0.4$, and the integration time step is $0.04/\omega_c$. The corresponding growth rate Γ_m is $0.023\omega_c$. The total energy change throughout the calculation was less than 0.2%.

The results of the computer experiment are shown in Figs. 1 to 3. In Fig. 1 evolution of the y (with z axis in the direction of the applied magnetic field) component of the vector potential of the wave field A_y is plotted as the solid curve. For illustrative purposes, the results are shown for two periods. The abscissa is the distance along z but shifted by $v_g t$, where v_g is the group velocity of the wave, so that the envelope stays fixed to the frame in the linear regime. The phase of the carrier wave which seems also fixed to the frame is accidental in that $(v_p - v_g)t$ at these particular times came out to be roughly

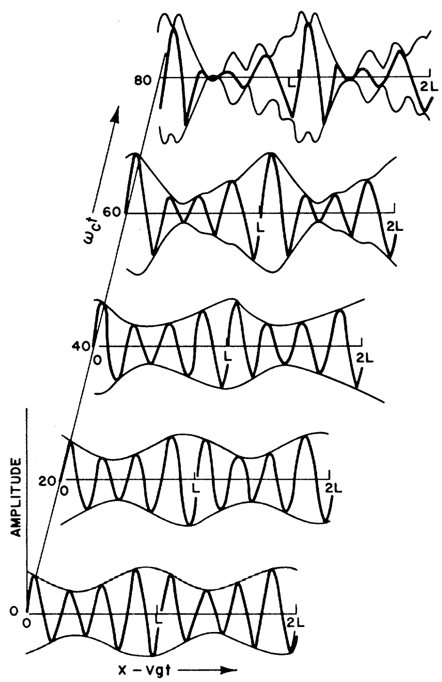

FIG. 1. Variation of wave amplitude A_y and energy $|A|$ with time and space; results of computer experiment.

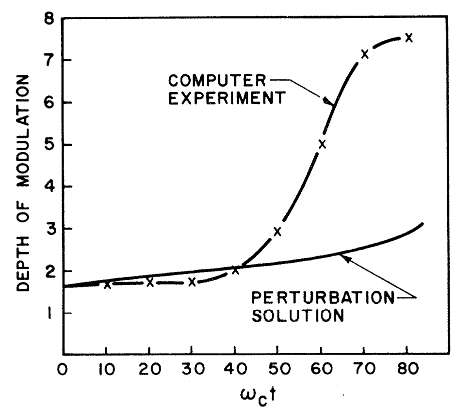

FIG. 2. Change of the depths of modulation. An explosive growth can be seen in the result of the computer experiment.

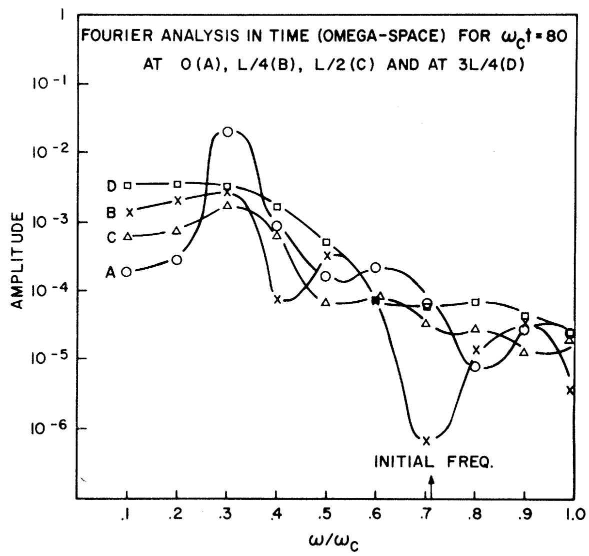

FIG. 3. Frequency spectrum of the wave at $\omega_c t=80$. The peak frequency is seen to have been shifted down.

a round number times the carrier wavelength. For the wave number chosen here, the phase velocity v_p is $1.8v_g$. The thin curve is $|A|=(A_y^2+A_z^2)^{1/2}$. Because the wave is circularly polarized in the linear regime, $|A|$ represents its envelope φ. But in the nonlinear regime $|A|$ starts to have a fine oscillatory structure indicating the loss of circular polarization.

We can see from this figure that the envelope first starts to become steeper in a way which is similar to the shock. At the same time, the envelope shows a gradual growth in amplitude. This continues until $\omega_c t \sim 40$. After this at $\omega_c t$ from 40 to 60, the amplitude of the envelope suddenly grows very fast, clearly indicating the process of the wave trapping. The rate of growth at this time is much faster than that given by the perturbation solution, Eq. (9). This can be most clearly seen in Fig. 2, where the depth of modulation defined as the ratio of the maximum to the minimum amplitude of the carrier wave is compared for the computer experiment with the perturbation solution given by Eqs. (6), (7), and (10). The slightly larger growth rate of the perturbation solution at initial time may originate from the use of only the growing-mode solution, while in the experiment both the growing and the decaying modes [negative imaginary Ω solution of Eq. (9)] should exist.

Another interesting feature of the result can be seen in the development in the frequency spectrum associated with the instability. As is shown in Fig. 3, at $\omega_c t=80$, the peak of the spectrum was shifted from its initial value of $0.72\omega_c$ to $0.3\omega_c$. The spectrum deviates rather remarkably at different points in space owing to the dispersive nonlinearity, but the average tendency is a shift toward lower frequencies. The peak frequency $0.3\omega_c$ is interesting because, as can be seen from Eq. (4), β changes its sign at $c^2k^2/\omega_p^2=\frac{1}{3}$ or at the corresponding frequency $\omega=0.25\omega_c$. Hence the frequency is down shifted toward the critical value for stability.[10]

We have performed another experiment with a smaller value of the initial depth of modulation ($\rho_1/\rho_0=0.2$) for an extended time scale. In this case, the time needed to reach to the explosive state was longer ($\omega_c t \sim 90$). Beyond this state, both the carrier wave and the modulation amplitude became jagged and the wave number of the carrier was shifted down. At $\omega_c t \sim 150$, the wave number, which was four per period at $t=0$, was clearly changed to two. No solitary wave shape was observed in the modulation amplitude throughout the process, unlike the previous prediction.[2] The fact that such a drastic change occurs for the carrier wave number and its frequency immediately after the explosive state indicates that the adiabatic approximation that has led to Eq. (1) becomes invalid in the same time scale as the variation of the amplitude φ itself. Hence the solution of Eq. (1) in its own time scale seems to be irrelevant as the answer of the development of the entire system. Right after the explosive state, the current sheets were suddenly accelerated in the longitudinal direction indicating a heating of the plasma. The gain of thermal motion caused the damping of the wave energy, presumably due to a cyclotron damping mechanism. The details will be published soon.

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[10]The peaking of the spectrum at $0.3\omega_c$ may not occur for the real ion-cyclotron wave because of the different dispersion relation, as is mentioned in Ref. 4. However, for the electron-cyclotron wave this effect should exist because of the same dispersion relation as assumed here.

ENERGY SPECTRUM OF A DILUTE HARD-SPHERE BOSE GAS*

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We use linear response theory to calculate the elementary excitation spectrum of a dilute hard-sphere Bose gas. Our result differs by a factor $2^{-1/2}$ from the exact result.

The energy spectrum of elementary excitations corresponding to density fluctuations of a many-particle system is given by the values of energy which correspond to the poles of a generalized susceptibility.[1] Recently Singwi et al.[2] proposed a self-consistent way of calculating the susceptibility of a many-electron system which seemed very promising in treating divergent potentials, but its use was restricted to systems in which the pair distribution function was known in advance.[3] The purpose of this Letter is to apply Singwi's theory to divergent potentials and derive the excitation spectrum from first principles, i.e., without the use of any parameter but the particle-particle potential and the density. We chose the worst possible potential, namely, the hard-sphere interaction, and we take the particles to form a dilute Bose gas.

As in Ref. 2 we take the susceptibility to be of the form

$$\chi(q\omega) = \frac{\chi_0(q,\omega)}{1-\psi(q)\chi_0(q,\omega)}, \tag{1}$$

where χ_0 is the noninteracting susceptibility. In the limit of small q, $\psi(q)$ is given by[4]

$$\psi(q) = -\left(\frac{4\pi}{3q^3}\right)\int_0^\infty dr g(r)[\sin(qr) - qr\cos(qr)]\left[\frac{d\varphi}{dr} - \left(\frac{r}{2}\right)\left(\frac{d^2\varphi}{dr^2}\right)\right]. \tag{2}$$

It is very easy to show that, at zero temperature, the relation between the energy of the excitation and $\psi(q)$ is, with $\epsilon(q) = \hbar^2q^2/2m$ and a density n,

$$E(q) = \{[\epsilon(q)]^2 + 2n\epsilon(q)\psi(q)\}^{1/2} = \epsilon(q)/S(q), \tag{3}$$

the well-known Feynman expression for bosons.[5] To calculate $\psi(q)$ we substitute in Eq. (2) the hard core for φ, putting

$$\frac{d\varphi}{dr} = \lim_{\lambda\to\infty}(-\lambda)\delta(r-r_c),$$

where r_c is the hard-core diam. We get for $\psi(q)$ the following expression:

$$\psi(q) = (2\pi/3q^3)\lambda[g(r_c)\alpha(qr_c) + r_c g'(r_c)\beta(qr_c)], \tag{4}$$

where

$$\alpha(x) = 3\sin x - 3x\cos x + x^2\sin x, \quad \beta(x) = \sin x - x\cos x.$$

We assume, at this point, that $g(r_c) = g'(r_c) = 0$. Thus, the products $\lambda g(r_c)$ and $\lambda g'(r_c)$ are not determined. To find out what their values are we calculate $g(r)$ and $g'(r)$ from $\psi(q)$,[6] and impose that $g(r_c) = g'(r_c) = 0$, consistent with our previous assumption.