THEORY OF AURORA BAND

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The predicted sheet current which connects the partial ring current and the auroral electrojet in auroral substorms is shown to be unstable against a shear perturbation (kink instability). A consequence of the instability is the deformation of the sheet current into a curtain shape as is visibly observed in auroras.

In a recent paper,[1] Akasofu and Meng have constructed a three-dimensional current system during a polar magnetic substorm based on their analyses of magnetic field data from more than 70 surface stations and those from several satellite stations. They have pointed out the significance of the field-aligned (parallel to geomagnetic field) sheet current with a magnitude of $\sim 10^6$ A that flows inward in the morning sector and outward from the evening sector. This current system explains the simultaneous variations of the geomagnetic field in the magnetosphere and at the earth's surface. Because the appearance of aurora is a manifestation of the polar magnetic substorm, such a current system should be closely related to the production and formation of aurora. Akasofu, in his succeeding paper,[2] has predicted that the ion sound-wave instability associated with the field-aligned current can cause electron acceleration and produce the aurora breakup.

A typical pattern of an aurora projected on earth is shown in Fig. 1. Two characteristics important to our interest are these: (1) The cross section of each aurora band A, B, C, has a width of a few thousand kilometers (east-west direction) and a thickness of several hundred meters (north-south direction).[3] (2) Each band has wavy patterns as indicated by a, b, c, with wavelengths of a few hundred kilometers.[4]

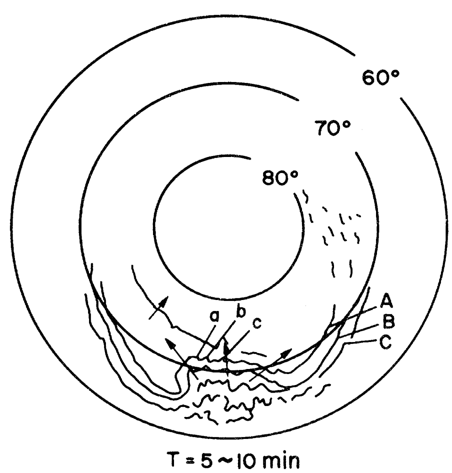

FIG. 1. Schematic illustration of the explosive phase of aurora substorm in polar plot. After Ref. 4.

The aurora itself is considered to be produced by the ionization due to suddenly increased medium energy (tens of kilovolts) electrons and protons. However, if, as is shown by Akasofu and Meng, the position of the field-aligned current does in fact coincide with that of the aurora, the implications are that the cross-sectional patterns of the aurora are also those of the current and that the ionizing particles follow the current path. In this paper we show that the field-aligned sheet current with the predicted dimensions and magnitude is unstable and deforms into a wavy shape in its cross section with wavelength corresponding to that of the observed aurora band. The instability is the kink instability that is caused by the tendency of the magnetic field, which is bent by the current, to become straight. We also discuss acceleration mechanisms for particles associated with the current.

The field-aligned current of interest has a shape as shown in Fig. 2. Because of the large plasma conductivity, the current is assumed to flow only at the surface. For the analysis we ignore the curvature and assume that the current-carrying plasma slab has a uniform cross section with a thickness $2a$, width $2b$, and length L. The validity of such assumptions will be justified later. We take the coordinate system as shown in Fig. 3. z is the direction of the current I_0 and the ambient magnetic field B_{0z}.

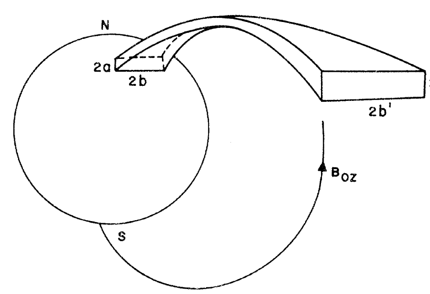

FIG. 2. Shape of current-carrying plasma slab.

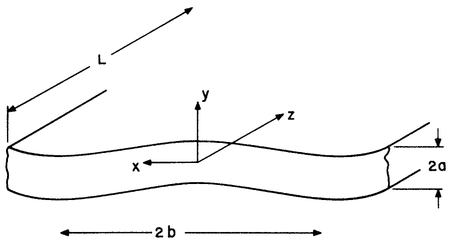

FIG. 3. Coordinate system as used in the theory.

Assuming a shear mode, the linearized magnetohydrodynamic equations lead to the following equation[5] for the displacement vector $\vec{\xi}$ and the z component of the perturbed magnetic field B_z:

$$(K_z^2-\omega^2/v_A^2)\vec{\xi}=-\nabla(B_z^{\,i}/B_{0z}), \tag{1}$$

where superscript i is used for the internal field; v_A is the Alfvén speed. The boundary conditions at $y=\pm a$ are (1) the conditions of the linearized pressure balance,

$$B_z^{\,i}B_{0z}=B_z^{\,e}B_{0z}+B_x^{\,e}B_{0x}^{\;e}+\frac{\xi_y}{2}\frac{\partial|B_0^{\,e}|^2}{\partial y}; \tag{2}$$

(2) the condition that the tangential electric field vanishes,

$$B_y^{\,e}=(\vec{B}_0^{\,e}\cdot\nabla)\xi_y. \tag{3}$$

Superscript e is used to indicate the external field. The surface current produces a jump in B_{0x} as indicated by Eq. (2).

Now it is well known that an infinitely extended sheet current ($b\to\infty$) is stable against a shear perturbation. This is because the external field $B_{0x}^{\;e}$ produced by such a current is constant, thus the last term in Eq. (2) vanishes. Namely the pressure balance is maintained independent of the size of the surface displacement ξ_y. On the other hand, if we consider a current with finite width, $B_{0x}^{\;e}$ decreases away from the current, i.e., $\partial|B_0^{\,e}|^2/\partial y<0$; hence, the corresponding term in Eq. (2) becomes effective. Then a small increase in ξ_y tends to break the pressure balance in such a way that the external pressure becomes weaker and thus ξ_y grows. When b is finite, the dc magnetic field B_0 becomes nonuniform, hence the exact treatment becomes complicated. However, if the thickness as well as the wavelength in the x direction are much smaller than b, the local treatment, such that every perturbation quantity be dependent on a form $f(y)\exp i(k_xx+k_zz-\omega t)$, is possible near the middle portion of the sheet in the x direction. There $\partial|B_0^{\,e}|^2/\partial y$ can also be shown to be constant and be equal to $\mp(4/\pi b)(B_{0x}^{\;e})^2$ at $y=\pm a$. It can then be shown that the asymmetric mode given by

$$\xi_y\sim\frac{\kappa}{\omega^2/v_A^2-k_z^2}\frac{\cosh(\kappa y)}{\sinh(\kappa a)} \tag{4}$$

is more unstable than the symmetric mode. This corresponds to the kind of perturbation shown in Fig. 3. For this mode, the dispersion relation is shown to have the following form,[6]

$$\frac{\omega^2}{v_A^2}=k_z^2+(k_z+\alpha k_x)^2\coth(\kappa a)-\frac{2\alpha^2}{\pi b}\kappa\coth(\kappa a), \tag{5}$$

where

$$\kappa=(k_x^2+k_z^2)^{1/2}, \tag{6}$$

and $\alpha(=B_{0x}^{\;e}/B_{0z})$ is the relative magnitude of the external magnetic field in the x direction produced by the dc current I_0, i.e.,

$$B_{0x}^{\;e}=\frac{\mu_0I_0}{2b}. \tag{7}$$

ω^2/v_A^2 can become negative because of the last term on the right-hand side of Eq. (5). The negative sign of this term arises from the negative gradient of $B_{0x}^{\;2}$ in the outward direction; that is, the instability arises owing to the fact that the external magnetic pressure decreases for an increased perturbation of the surface. ω^2/v_A^2 becomes minimum when

$$k_z+\alpha k_x=0, \tag{8}$$

and the condition of instability is

$$k_z<[(2\kappa/\pi b)\coth(\kappa a)]^{1/2}\alpha,$$

or because $\kappa a\ll 1$,

$$k_z<(2/\pi ab)^{1/2}B_{0x}^{\;e}/B_{0z}. \tag{9}$$

If we take k_z to be π/L, Eq. (9) becomes

$$B_{0x}^{\;e}/B_{0z}>(\pi^3/2)^{1/2}(ab)^{1/2}/L. \tag{9a}$$

Equation (9a) corresponds to the Kruskal-Shafranov limit of the kink instability for a slab geometry. If we use I_0, this condition can also be expressed as

$$I_0>\frac{2b}{\mu_0}\left(\frac{\pi^3}{2}\right)^{1/2}\frac{(ab)^{1/2}}{L}B_{0z}. \tag{9b}$$

If a, b, and B_{0z} vary along the z direction as shown in Fig. 2, flux conservation gives

$$4B_{0z}ab=\psi_0=\text{const};$$

hence Eq. (9b) becomes

$$I_0 > \frac{2\psi_0}{\mu_0}\left(\frac{\pi^3}{2}\right)^{1/2}\frac{1}{L}\left(\frac{b}{a}\right)^{1/2}. \tag{9c}$$

Therefore so long as b/a remains constant the instability condition becomes independent of z. For example, if we take the value at the earth's surface, $a=0.5$, $b=2\times10^3$ km, $B_{0z}=5\times10^4\ \gamma$, and $L=1.5\times5R_E=4.5\times10^4$ km (R_E= earth's radius), Eq. (9a) gives

$$B_{0x} > 4\frac{(ab)^{1/2}}{L}B_{0z} = 3\times10^{-3}B_{0z} = 150\ \gamma\ \ (1\ \gamma = 10^{-5}\ \text{G}).$$

The typical magnetic field disturbance during the polar substorm is 300-500 γ, thus the above condition seems to be well satisfied. In terms of the current I_0, the instability condition becomes, from Eq. (9b),

$$I_0 > 4.5\times10^5\ \text{A},$$

which is again satisfied with the observed value of 10^6 A.

The wavelength of the instability in the x direction at the threshold is obtained from Eqs. (8) and (9) as

$$\lambda = \frac{2\pi}{k_x} = 2\pi\left(\frac{\pi ab}{2}\right)^{1/2} \sim 8(ab)^{1/2}. \tag{10}$$

If we use the same data,

$$\lambda \sim 240\ \text{km}.$$

This wavelength again is the typical observed value for the aurora curtains.

The curvature of the field lines makes an additional contribution to $\partial|B_0^e|^2/\partial y$, but the additional part, which is $\sim L^{-1}$, is negligible compared with the original gradient, that is $\sim b^{-1}$, because $L \gg b$. However, the curvature drift of the current-carrying plasma can cause a time-increasing charge separation in the x (east-west) direction, which produces $\vec{E}\times\vec{B}$ acceleration of the current toward the pole. Such an acceleration, which corresponds to a flute mode, may be the cause of the rapid poleward motion of the aurora.

Acceleration of ionizing particles has been considered to take place at the tail region of the magnetosphere by many authors.[7] However, because of evidence that the acceleration takes place rather locally, Swift[8] suggested that it may be produced by the ion sound-wave instability due to the current-carrying electrons. Kennel suggested the ion-cyclotron instability as an alternative possibility.[9] However, the present instability may also produce the acceleration of these ionizing particles because in the nonlinear stage of the instability the field-aligned discharge current may be disrupted and will produce a large voltage drop along the field line. The fairly good agreement between the critical current obtained here and that observed for the aurora breakup seems to support this possibility.

If a microinstability is produced by the surface current, the finite conductivity which may be produced by it induces current to diffuse inward. Under such circumstances, the present analysis fails and a more complicated treatment including the distributed current density is required. However, as was shown by Suydam[10] for a cylindrical case, a distributed current system is also unstable under similar conditions provided B_{0z} is constant, hence qualitatively the present result should still apply.

Finally, the sheet current as assumed here is not in pressure-balance equilibrium at the corners. Hence the slab form may be gradually changed to a cylindrical shape, which is the equilibrium shape, even under the stable conditions. Such a change occurs starting from the corner ($x=\pm b$) at a speed equal to the nonlinear Alfvén velocity v_A' given by $v_A' = 7.4\times10^{-5}cB_{0x}^e/n^{1/2}$ m/sec, where n is the number density in cm^{-3}, c is the speed of light in m/sec, and B_{0x} in gamma. The time τ needed to become a cylindrical form is then given by $\tau \sim b/v_A' \sim 200$ sec, for $B_{0x}^e = 150\ \gamma$, and $n=10^5\ \text{cm}^{-3}$.

On the other hand, the growth time of the instability, τ', can be obtained from Eq. (5) as

$$\tau' \sim (\tfrac{1}{2}\pi ab)^{1/2}/v_A' \sim 4\ \text{sec} \ll \tau.$$

Hence the curtain shape is produced much faster than the total deformation of the sheet into a cylinder. However, because 200 sec is not large compared with the storm period, the large-scale change can also be seen during the aurora development. Large bulges observed at the edge of the aurora may be due to this process.

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OBSERVATION OF SELF-TRAPPING INSTABILITY OF A PLASMA CYCLOTRON WAVE IN A COMPUTER EXPERIMENT

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The exact behavior of a modulated cyclotron wave in a plasma is traced in a numerical model to study the self-trapping instability represented by the nonlinear Schrödinger equation with an attractive potential. The instability is found to be explosive in that the wave collapses at a much faster rate than the prediction of the perturbation solution.

In a recent paper,[1] Taniuti and Washimi have shown that a modulated whistler wave becomes unstable owing to its self-action produced by the third-order nonlinearity. They derived a nonlinear Schrödinger equation for the modulation amplitude $\varphi(x,t)$ of a whistler wave represented by $\varphi(x,t)\exp[i(kx-\omega t)]$ and showed that the equivalent potential term is attractive in sign.

If a nonlinear dispersion characteristic $\omega(k,a^2)$ is known, where a is the amplitude of the wave, the modulation amplitude of a finite amplitude wave φ in the nonlinear dispersive medium can generally be expressed by the nonlinear Schrödinger equation of the following form[2]:

$$i\frac{\partial\varphi}{\partial t}+\alpha(|\varphi|^2-|\varphi_0|^2)\varphi+\beta\frac{\partial^2\varphi}{\partial x^2}=0, \qquad (1)$$

where $\alpha=-\partial\omega/\partial a^2$, $\beta=\frac{1}{2}\partial^2\omega/\partial k^2$, and φ_0 is the initial value of φ. The condition of self-trapping can easily be derived from Eq. (1) as the condition of attractive potential,

$$(\partial\omega/\partial a^2)(\partial^2\omega/\partial k^2)<0. \qquad (2)$$

Using this concept Sivasubramanian and Tang[3] have derived the unstable region in the dispersion relation for waves in a cold magnetized plasma. Hasegawa[4] has shown that the sign of the $\partial\omega/\partial a^2$ term, which is crucial in deciding the stability, can be changed arbitrarily in the presence of a coupling to a nondispersive low-frequency mode. Tam[5] and Petviashvile,[6] using somewhat different approaches, have shown, respectively, that obliquely (with respect to magnetic field) propagating whistler and ion-acoustic waves face similar instabilities.

A mathematical solution of Eq. (1) for an attractive potential is not available, however, so the actual time evolution of the instability is still unknown. Because there is a large class of physical problems[7] that are representable by the conspicuous characteristic of the nonlinear Schrödinger equation, it is of great interest to study the dynamic evolution of Eq. (1). In this paper instead of solving Eq. (1), we discuss the results of computer experiments on a dynamic system whose long-time asymptotic behavior is represented by this equation. Thus we can obtain the dynamic behavior not only of the modulation amplitude but also of the wave itself in such a system. The problem we treat is a transverse perturbation of ions propagating parallel to an applied magnetic field in a cold plasma. In the linear regime the perturbation corresponds to the ion-cyclotron wave (this can also be regarded as an electron-cyclotron wave by changing the polarization and the time constant). We use the sheet-current model[8] in which the transverse motion of the charged particles is represented by a set of infinite sheet currents arranged perpendicular to the applied magnetic field.

In this case, the coefficients of Eq. (1) can be shown to be[4]

$$\alpha=-\frac{\omega_c}{8}\left(1+\frac{c^2k^2}{\omega_p{}^2}\right)^2, \qquad (3)$$

$$\beta=\frac{c^2\omega_c}{\omega_p{}^2}\frac{1-3c^2k^2/\omega_p{}^2}{(1+c^2k^2/\omega_p{}^2)^3}, \qquad (4)$$