In conclusion, our findings demonstrate that a trapped particle can maintain phase coherence for a long time, and its anharmonic orbit can be responsible for the new echoes observed. It is possible that "fractional echoes" also occur in other plasma[1] and solid-state systems[4] but have escaped detection because the sensitive detectors of these experiments operating at higher frequencies cannot respond to a wide range of frequencies. Effects of local fields can be precisely measured by this temporal echo method, in contrast to the spatial echo technique[5] which averages the effect of fluctuations over a certain distance. Although the magnetic field does not play a dominant role in the foregoing experiments on account of the large ion Larmor radius ($\rho_i \approx 0.2$ cm), we are presently investigating the effects of a small Larmor radius ($\rho_i \approx 10\lambda_D$) on trapped particle orbits. Our experiments suggest investigations of particles trapped inside magnetic inhomogeneities or in strong electromagnetic fields.[6] The present experimental arrangement is probably the simplest in which echoes can be observed, requiring no stringent restrictions on either magnetic field or density inhomogeneities. The long echo lifetime of approximately 1 msec permits observation of both low- and high-frequency fluctuations. The simple new ion probe technique, which is capable of rather precise measurements, should find interesting applications in space and laboratory plasmas.

We wish to acknowledge the able assistance of C. Y. Liu in computing various Fourier coefficients.

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*Research supported by Department funds and by the U. S. Air Force Office of Scientific Research under Grant No. 962-67.

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EFFECT OF INERTIA ON LOSSES FROM A PLASMA IN TOROIDAL EQUILIBRIUM

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(Received 24 July 1969)

We investigate the magnetohydrodynamic equilibrium of a resistive, low-density plasma in a model stellarator field. The effect of inertia on plasma motion is treated exactly, and its influence on plasma loss determined. It is shown that the losses due to inertia are limited by the conditions for the existence of an equilibrium.

The plasma loss rate from a low-density resistive plasma in a torus has previously been calculated[1,2] neglecting finite plasma inertia. In other papers[3,4] the effect of inertia was taken into account, but in such a manner that the effect on the resistive losses was contained in an order of calculation higher than the one considered. Recently, the problem has been attacked by Stringer,[5] who described the plasma by a mixture of guiding-center and fluid equations, which he solved in the limit of large aspect ratio. The main result of his calculation is the apparently unlimited increase over the Pfirsch-Schlüter loss rate, whenever a certain resonance condition is satisfied.

On the other hand, our treatment of the same problem is valid for arbitrary aspect ratio, and includes the inertia term exactly. By discussing the requirements for a stationary solution, we obtain restrictions on plasma flow, from which we argue that the increase in plasma loss due to inertia is also limited. This conclusion appears to be in agreement with the preliminary results of a numerical treatment of the problem.[6]

The equations we use are the well-known fluid equations

$$\rho(\tfrac{1}{2}\nabla v^2 - \vec{v}\times \mathrm{rot}\vec{v}) = \vec{J}\times\vec{B} - \nabla p, \tag{1}$$

$$\vec{v}\times\vec{B} = \eta\vec{J} + \nabla\varphi, \tag{2}$$

$$\mathrm{div}\rho\vec{v} = Q, \tag{3}$$

$$\mathrm{div}\vec{J} = 0, \tag{4}$$

where the symbols have their usual meaning. We consider a low-density plasma, so that $\vec{B}$ is a given external magnetic field. Q, the plasma source distribution, is also assumed given as a function in space. Further, plasma is taken to be injected with the local fluid velocity $\vec{v}$. Although it is possible to be more general, we refer here to an isothermal plasma (constant sound speed c).

For the magnetic field we employ the model[1]

$$\vec{B} = B_0 R[(f\kappa/N)\nabla\theta + \nabla\xi], \tag{5}$$

where

$$N = 1 + \kappa\cos\theta, \quad \kappa = r/R, \quad f = \iota(r)\kappa/(1-\kappa^2)^{1/2}. \tag{6}$$

The r, θ, ξ co-ordinate system has been described previously.[3] Important parameters are κ, the inverse of the aspect ratio, and ι, the usual rotational transform divided by 2π.

We now describe our solution procedure for Eqs. (1) to (4) together with the isothermal plasma assumption. An axisymmetric solution is sought. We make an expansion in resistive effects, so that in zeroth order we treat ideal plasma flows, with inertia fully included. This means that already in the first order of the expansion, resistive plasma losses (modified by inertia) will appear. There is a relationship between the successive orders of expansion so that the arbitrariness in each order is removed by a certain physical requirement on the solution in the next higher order. It can be shown that solving consistently, order by order, leads to a unique solution. The details of this calculation will appear elsewhere.[7]

The zeroth-order solution can be expressed in terms of arbitrary surface functions (functions constant on a magnetic surface), which in turn are determined by the conditions arising from restrictions on the first-order solution. Of particular use in a description are the mass fluxes the long and the short way, given by $2\pi\psi$ and $2\pi\Gamma$, respectively:

$$\psi = \int_0^r \langle (g)^{1/2}\rho\vec{v}\cdot\nabla\xi\rangle dr', \tag{7}$$

$$\Gamma = \int_0^r (g)^{1/2}\rho\vec{v}\cdot\nabla\theta dr'. \tag{8}$$

Here g is the determinant of the metric tensor of the coordinate system. Averages over the azimuthal angle are denoted by $\langle\,\rangle$. The equation governing the density variation on a magnetic surface, in zeroth order, is

$$u\ln u = [M^2 + E^2(N^2-1)]u - M^2/N^2, \tag{9}$$

where

$$M = \frac{\Gamma'(1-\kappa^2)^{1/2}}{\iota\gamma\rho_0 cr}, \quad E = \frac{\Gamma' - \iota\psi'}{\iota\rho_0 cr\langle N\sqrt{u}\rangle}, \tag{10}$$

and

$$u = \rho^2/\rho_0^2, \quad \rho_0(r) = \rho(r_1\tfrac{1}{2}\pi), \quad \gamma^2 = (1+f^2)^{-1}. \tag{11}$$

The restrictions imposed by the equilibrium solution arise from the physical requirement that Eq. (9) have a real solution u, for all θ. It can be shown that this is equivalent to

$$0 \leq E^2 \leq \frac{\ln\{(e^{M^2-1}/M^2)(1-\kappa)^2\}}{1-(1-\kappa)^2}. \tag{12}$$

Clearly the class of mass fluxes consistent with stationary equilibrium is restricted. This limitation on E and M is displayed in Fig. 1.

One feature to note is that there exist two allowed regions [given by Eq. (12)] which are not connected, in the sense that we cannot proceed from one to the other by a continuous change of mass fluxes. The region containing the origin of the M^2-E^2 plane includes the static case ($M = E = 0$). For this reason we exclude the other region from our consideration. For $M < 1$, Fig. 2 illustrates how a solution of Eq. (9) is obtained.

It is then clear that condition (12) comes from the requirement that for a real solution of u for all θ the slope of the straight line given by $\theta = \pi$ must be greater than the slope of the tangent line

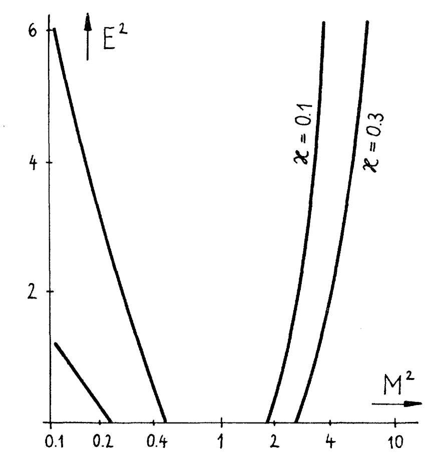

FIG. 1. Regions of E and M allowed by zeroth-order solution.

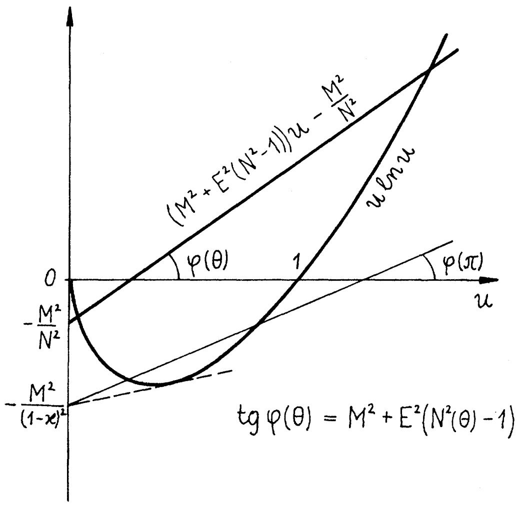

FIG. 2. Solution procedure for zeroth-order density distribution on magnetic surfaces.

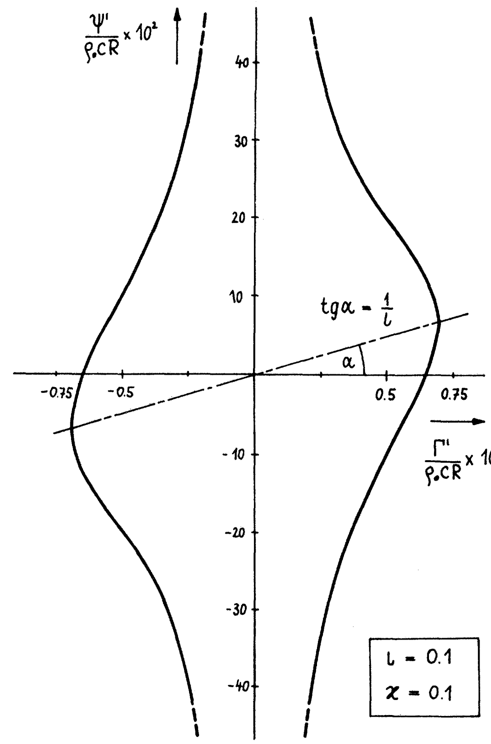

FIG. 3. Mass fluxes consistent with stationary equilibrium.

to $u\ln u$ having the same intercept on the vertical axis. In Fig. 3 this restriction is exhibited in terms of the r derivatives of the mass fluxes.

Clearly as $\iota \to 0$, the range of mass flux the short way that is consistent with a stationary solution becomes more strongly limited. This means that for azimuthal plasma rotation exceeding a certain value a stationary state is no longer possible. Such restrictions can be re-expressed in terms of the velocity components v_θ and v_ξ themselves, a fact which we use later in estimating the magnitude of the plasma mass-loss rate $W^{(1)}$. In the first order of finite-resistivity effects we obtain, for the case of a plasma source Q localized at $r=0$,

$$W^{(1)} = -\eta\frac{rRc^2}{B_0^2 f^2}\Big\{\Big\langle\rho\frac{\partial\rho}{\partial r}N^3\Big\rangle - \gamma^2\langle N^{-1}\rangle^{-1}\langle\rho N\rangle\Big\langle N\frac{\partial\rho}{\partial r}\Big\rangle - \frac{1}{r}\Big\langle\rho^2N^3\Big(\frac{v_\theta^2}{c^2}+\kappa\frac{\cos\theta}{N}\frac{v_\xi^2}{c^2}\Big)\Big\rangle + \frac{1}{r}\gamma^2\langle N^{-1}\rangle^{-1}\langle\rho N\rangle$$
$$\times\Big\langle\rho N\Big(\frac{v_\theta^2}{c^2}+\kappa\frac{\cos\theta}{N}\frac{v_\xi^2}{c^2}\Big)\Big\rangle + \Big\langle\rho N\frac{\partial}{\partial r}\Big(\frac{1}{f}\rho N^2\frac{v_\theta}{c}\frac{v_\xi}{c}\Big)\Big\rangle - \langle N^{-1}\rangle^{-1}\langle\rho N\rangle\Big\langle\frac{1}{N}\frac{\partial}{\partial r}\Big(\frac{1}{f}\rho N^2\frac{v_\theta}{c}\frac{v_\xi}{c}\Big)\Big\rangle\Big\}. \quad (13)$$

The various terms appearing in Eq. (13) can be understood as follows. The first bracket is very similar to the usual resistive diffusion term, and in fact when inertia is neglected, i.e., when the density variation on magnetic surfaces vanishes, it is exactly that term. The remaining terms represent the main modification introduced by the inclusion of plasma inertia. The terms involving

$$a_r \equiv (\vec{v}\cdot\nabla\vec{v})\cdot\nabla r = -\frac{1}{r}\Big(v_\theta^2+\kappa\frac{\cos\theta}{N}v_\xi^2\Big),$$

the r component of the local acceleration, are related to the centripetal force, which results mainly from toroidal flow. This force must be partially balanced by the appropriate $\vec{J}\times\vec{B}$ component. Part of the finite electric currents in a resistive plasma represent the attempt of the plasma to short out the electrostatic fields developed in the flow, and which, together with the magnetic field, move plasma across magnetic surfaces. The remaining part of the electric current combines with the magnetic field to confine the plasma. The final bracket is related to a Coriolis-type force in the ξ direction. Such a force must be totally balanced by a $\vec{J}\times\vec{B}$ force, as the assumption of axisymmetry removes the possibility of a pressure gradient in this direction.

Now the velocity components in zeroth order are given by

$$\frac{v_\theta}{c}=f\frac{\gamma M}{N\sqrt{u}}, \tag{14}$$

$$\frac{v_\xi}{c}=\frac{\gamma M}{N\sqrt{u}}-NE. \tag{15}$$

Thus it can be seen that while the first bracket in Eq. (13) varies as ι^{-2} for small ι, the other terms involve a ι^{-4} dependence. Formally it would appear that for small ι we do have a large increase in plasma loss over the Pfirsch-Schlüter result. However, in the expansion procedure, we have still to determine the mass fluxes exactly. This is done by requiring that the solution in first order be periodic in θ. Closer examination reveals that from this we obtain two complicated differential equations with respect to r, for Γ and ψ. The solution of these equations with the appropriate boundary conditions would enable us to evaluate $W^{(1)}$ exactly. Rather than carry out this full prescription, we have made the plausible assumption that the zeroth-order angular momentum through each magnetic surface is zero. Such an assumption relates M and E:

$$\gamma M=E\langle N^3\sqrt{u}\rangle. \tag{16}$$

This is in addition to the zeroth-order limitations imposed on Γ and ψ which were mentioned above. Equation (16) and the solvability condition (12) lead directly to restrictions on the components of flow velocity:

$$|v_\theta/c|\leq f\gamma, \tag{17}$$

$$\left|\frac{v_\xi}{c}\right|\leq\gamma\left[1-\frac{1-\kappa}{\langle N^3\rangle}|M|\exp\left\{-\tfrac{1}{2}M^2\left(1+\frac{\gamma^2}{\langle N^3\sqrt{u}\rangle}[(1+\kappa)^2-1]\right)\right\}\right]\leq 1, \tag{18}$$

where for representative ι, κ values, $\langle N^3\sqrt{u}\rangle\sim 1$.

One further assumption is required before we can estimate the orders of magnitude of the different terms in expression (13). This concerns the "smoothness" of the radial variation of quantities. We require that all functions be well behaved in the sense that $\partial A/\partial r\sim A/r$ is a reasonable approximation.

With this information it is easy to show that each of the three brackets in expression (13) is of the same order of magnitude. In other words the contributions to plasma loss due to inertia are, at most, several times the value of the term describing classical diffusion.[1] This result seems to agree with preliminary calculations involving the numerical integration of the fluid equations. For representative situations, a total plasma loss of approximately five times the Pfirsch-Schlüter result is found.

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REVERSIBILITY AND THE DAMPING OF A DOUBLE BEAM INSTABILITY*

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(Received 17 January 1969; revised manuscript received 24 September 1969)

An experimental and theoretical investigation has been made of the absolute instability that occurs near $\frac{1}{2}\omega_c$ when two opposing drifting electron beams interpenetrate in a static magnetic field. The theory allows determination of the beam temperature by measurement of cutoff magnetic field or cutoff beam velocity. The result of the measurement is that the beam temperature is equal to the cathode temperature and we conclude that no irreversible change takes place.

It is well known that the Vlasov equation describes reversible phenomena in a collisionless plasma and it has been decisively proved by Malmberg et al.[1] by their echo experiment that under proper experimental conditions the information left behind after phase mixing can be recovered by application of a second signal. This implies that the temperature of interacting beams which give rise to a growing instability should not change. On the other hand Etievant and Per-