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RIGOROUS SOLUTION OF ELECTRON TRANSPORT PHENOMENA IN GASES

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A rigorous method is presented to obtain the velocity distribution function $F(c,\theta)$ for free electrons in gases in quasistationary and quasiuniform cases. It is based on the introduction of a particular distribution function $f_0(c_0)$ relevant to the "initial" speeds of successive paths taken by an electron. The "usual" distribution function $F(c,\theta)$ can then be obtained from the "initial" distribution function $f_0(c_0)$ by an integration.

In the presence of a quasiconstant (with respect to the relaxation times) and quasiuniform (with respect to mean free paths) electric field $\vec{E}$, the distribution function of electrons in a gas may be represented by[1] $F^*(c, \theta, \vec{r}) = F(c, \theta)\varphi(\vec{r})$, where c is the magnitude of the velocity, θ is the angle between $\vec{c}$ and $\vec{E}$, and $\vec{r}$ the position vector. However, the velocity distribution function $F(c, \theta)$ has never been derived, since Boltzmann's integrodifferential equation is practically intractable in the case of a function of two variables. Therefore all authors expand $F(c, \theta)$ in Legendre polynomials[2] of $\cos\theta$ or in some other way[3] in order to have for the unknown a function $f(c)$ of a single variable. Moreover they do not retain the entire series but usually the first-order terms only. This is equivalent to the following two simplifying assumptions, corresponding to a small anisotropy of $F(c, \theta)$: (i) The velocity variations Δc_E between two successive collisions, due to the accelerating field $\vec{E}$, are small with respect to the thermal velocity c; (ii) the inelastic collision frequency $\nu_{\rm in}$ is small with respect to the elastic collision frequency $\nu_{\rm el}$.

In a previous paper,[4] a particular distribution function relevant to the "initial" velocities of successive paths taken by an electron has been introduced. This "initial" distribution function, being unaffected by the electric field, is isotropic when the differential collision cross section is isotropic, and therefore it can be split into the following product:

$$F_0(c_0, \theta_0) = f_0(c_0)\pi_0(\theta_0) = f_0(c_0)\tfrac{1}{2}\sin\theta_0. \tag{1}$$

The resulting integral equation for $f_0(c_0)$, though rigorous, has the same simplicity as the usual first-order expansion of the Boltzmann equation in Legendre polynomials. It is[4]

$$f_0(c_0') = \int_0^\infty f_0(c_0)dc_0 \int_0^\pi \tfrac{1}{2}\sin\theta_0 d\theta_0 \int_0^\infty G(c \to c_0')Q(c_0, \theta_0, t)dt, \tag{2}$$

where $Q(c_0, \theta_0, t)$ is the distribution of flight times [given by Eq. (23) of Ref. 4] and $G(c \to c_0')$ is the probability density for speed changing from c to the new initial velocity c_0' upon scattering [given in Ref. 4 by Eq. (35) for inelastic collisions and by Eq. (36) for elastic collisions]. Equation (2) has been recently solved in the case of elastic collisions and constant flight times.[5]

By means of $f_0(c_0)$, the rigorous expression of the drift velocity W can be obtained[4,6] as the ratio between the total displacement (in the direction of the field) of an electron (proportional to the mean displacement) and the total time of flight (proportional to the mean time of flight T):

$$W=\frac{S}{T}=\frac{\int_0^\infty f_0(c_0)dc_0\int_0^\pi \sin\theta_0 d\theta_0 \int_0^\infty sQ(c_0,\theta_0,t)dt}{\int_0^\infty f_0(c_0)dc_0\int_0^\pi \sin\theta_0 d\theta_0 \int_0^\infty tQ(c_0,\theta_0,t)dt}=\frac{\int_0^\infty f_0(c_0)w(c_0)T(c_0)dc_0}{\int_0^\infty f_0(c_0)T(c_0)dc_0}, \tag{3}$$

where $s=c_0t\cos\theta_0+\frac{1}{2}at^2$ (with $a=eE/m$ as electron acceleration) is the electron displacement in the $\vec{E}$ direction during a flight time t, $w(c_0)$ is the drift velocity for monoenergetic electrons having initial speed c_0 [the expression for $w(c_0)$ is given by Eq. (24) of Ref. 4] and $T(c_0)$ is the relevant mean flight time [given by the denominator of Eq. (24) of Ref. 4]. Notice that, in the case of constant flight times $\lambda/c=\tau_0$, it turns out that $T=T(c_0)=\tau_0$. Therefore, since in this case[4] $w(c_0)=a\tau_0$, one finds from Eq. (3) that $W=w(c_0)=a\tau_0$. This result extends the validity of the known expression for $\lambda/c=$constant to the case of high fields, i.e., when assumption (i) is no longer retained.

From the initial distribution function $f_0(c_0)$ of the speeds, it is possible not only to obtain a rigorous expression for the drift velocity, i.e., Eq. (3), but also to solve rigorously transport problems for electrons in gases when the electric field is slowly varying in time (with respect to relaxation times) and in space (with respect to mean free paths). Namely, let us show that it is possible to obtain the usual velocity distribution function $F(c,\theta)$ by a simple integration of the initial speed distribution function $f_0(c_0)$.

Consider a number d^3n_0 per unit volume of electrons "born" in the time interval dt and having initial velocities with magnitudes between c_0 and c_0+dc_0 and directions within the solid angle $d\Omega=2\pi\sin\theta_0 d\theta_0$. When the collision cross section is isotropic, d^3n_0 is given, using Eq. (1), by

$$d^3n_0=q(c_0,t)dtdc_0\tfrac{1}{2}\sin\theta_0 d\theta_0, \tag{4}$$

where the source term $q(c_0,t)$ is related to $f_0(c_0)$ by

$$f_0(c_0)=\int_0^T q(c_0,t)dt, \tag{5}$$

in which T is a convenient mean time in order to have a normalized $f_0(c_0)$. If $q(c_0,t)$ is appreciably constant during T (electric field slowly varying in time), Eq. (5) reduces to

$$f_0(c_0)=q(c_0,t)T, \tag{6}$$

and therefore Eq. (4) becomes

$$d^3n_0=f_0(c_0)dc_0\tfrac{1}{2}\sin\theta_0 d\theta_0 dt/T. \tag{7}$$

After a flight time t, a number of electrons d^3n_t remains, connected to the initial number d^3n_0 by Eq. (21) of Ref. 4:

$$d^3n_t=d^3n_0\exp[-\int_0^t c(\tau)/\lambda(\tau)d\tau]. \tag{8}$$

Let us now assume that the velocity of the above electron group (of number d^3n_t) is between c and $c+dc$ and between θ and $\theta+d\theta$. Since, during a free flight, the component of the motion in the direction perpendicular to acceleration $\vec{a}=e\vec{E}/m$ is uniform and the component parallel to $\vec{a}$ is uniformly accelerated, the initial speed c_0 and the initial direction θ_0 of the group considered are related to c and θ by the relationships

$$\begin{aligned} &c_0\sin\theta_0=c\sin\theta,\\ &c_0\cos\theta_0+at=c\cos\theta, \end{aligned} \tag{9}$$

from which

$$\begin{aligned} &c_0=(c^2+a^2t^2-2cat\cos\theta)^{1/2},\\ &\theta_0=\arctan\frac{c\sin\theta}{c\cos\theta-at}. \end{aligned} \tag{10}$$

Let us change the variables in Eq. (8), substituting in it Eq. (10) and taking into account that

$$dc_0d\theta_0=dcd\theta\begin{vmatrix}\partial c_0/\partial c & \partial c_0/\partial\theta\\ \partial\theta_0/\partial c & \partial\theta_0/\partial\theta\end{vmatrix}=dcd\theta\frac{c}{(c^2+a^2t^2-2cat\cos\theta)^{1/2}}.$$

We get

$$d^3n_t = f_0[(c^2+a^2t^2-2cat\cos\theta)^{1/2}]\frac{1}{2}\frac{c^2\sin\theta}{c^2+a^2t^2-2cat\cos\theta}\exp\left[-\int_0^t \frac{c(\tau)}{\lambda(\tau)d\tau}\right]\frac{dcd\theta dt}{T}. \tag{11}$$

"Summing" Eq. (11) for all the times of flights, i.e., integrating Eq. (11) between 0 and ∞, we get

$$\frac{d^2n}{dcd\theta} = F(c,\theta) = \frac{c^2\sin\theta}{2T}\int_0^\infty \frac{f_0[(c^2+a^2t^2-2cat\cos\theta)^{1/2}]}{c^2+a^2t^2-2cat\cos\theta}\exp\left[-\int_0^t\frac{c(\tau)}{\lambda(\tau)d\tau}\right]dt. \tag{12}$$

When the simplifying assumption (i) is valid, which is equivalent to the statement $at \ll c \simeq c_0$, Eq. (12) reduces to

$$F(c,\theta) = \frac{\sin\theta}{2T}f_0(c)\left|-\frac{\lambda(t)}{c(t)}\exp\left(-\int_0^t\frac{c(\tau)}{\lambda(\tau)}d\tau\right)\right|_{t=0}^{t=\infty} = \tfrac{1}{2}f_0(c)\sin\theta\frac{\lambda(c)/c}{T} = f_0(c)\,\tfrac{1}{2}\sin\theta\frac{T(c)}{T}.$$

Moreover

$$f(c) = \int_0^\pi F(c,\theta)d\theta = f_0(c)[T(c)/T], \tag{13}$$

which is the relationship intuitively postulated by Braglia,[7] based on the consideration that the probability of finding electrons started with a velocity $c_0 \simeq c$ is proportional to the relevant time of flight.

In order to have the usual normalized distribution function $F(c,\theta)$, the constant time interval T must be the mean time of flight given by the denominator of Eq. (3). This condition appears evident if one normalizes both sides of Eq. (13).

Summarizing, the method suggested here for solving problems of transport phenomena for electrons in gases (diffusion, drift, excitation, ionization, and so on) when the electric field is slowly varying in space and time, is the following one. One first solves the integral equation (2), in which the unknown is a function $f_0(c_0)$ of a single variable, and then obtains the usual velocity distribution function by an integration, as given by Eq. (12).

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