*Work supported in part by the U. S. Atomic Energy Commission Contract No. AT(45)-1388.

[1]G. Veneziano, Nuovo Cimento 57A, 190 (1968).

[2]For simplicity we are taking the t and u channels as identical.

[3]An infinity of different forms comes immediately to mind. Notice, for instance, that in order to satisfy unitarity exactly, ρ should depend on s and t to introduce double spectral functions. Also in nonsymmetric cases one could expect to smear separately in s and t, etc. It is hoped that as $\epsilon \to 0$, our form retains the important features.

[4]C. Lovelace, Phys. Letters 28B, 269 (1968).

[5]M. Ademollo, G. Veneziano, and S. Weinberg, Phys. Rev. Letters 22, 83 (1969).

[6]D. D. Coon, Phys. Rev. (to be published).

[7]H. Harari, Phys. Rev. Letters 20, 1395 (1968).

[8]J. Shapiro and J. Yellin, University of California Radiation Laboratory Report No. UCRL 18500 (to be published).

UNITARIZATION OF THE VENEZIANO MODEL, POMERANCHUK SINGULARITY, AND THE PION MASS*

Kerson Huang
Laboratory for Nuclear Science and Physics Department,
Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 19 May 1969)

We suggest a general framework for unitarizing the Veneziano model for $\pi\pi$ scattering, and deduce some immediate consequences. By duality the unitarity s cut generates a moving Regge cut with branch-point trajectory $1+\alpha' t$ in the approximation $m_\pi = 0$, with α' the slope of the ρ-f^0 trajectory. In principle m_π may be calculated from the shift of the Adler zero. A crude estimate yields $m_\pi \approx (\alpha')^{1/2} m_\rho \Gamma_\rho = 95$ MeV.

We suggest a general framework in which the unitarization of the Veneziano model[1] might be attempted, and describe some immediate qualitative consequences. Consider $\pi^+\pi^-$ scattering in the approximation of vanishing pion mass. The scattering amplitude in the Veneziano model is given by[2]

$$A(s,t) \approx V_0(\alpha_s, \alpha_t),$$

$$V_0(\alpha_s, \alpha_t) = \frac{-\beta\Gamma(1-\alpha_s)\Gamma(1-\alpha_t)}{\Gamma(1-\alpha_s-\alpha_t)} = \beta(1-\alpha_s-\alpha_t)\sum_{n=0}^{\infty}\binom{\alpha_t+n-1}{n}\frac{1}{\alpha_s-n-1}, \tag{1}$$

where $\alpha_t = \frac{1}{2}+\alpha' t$ is the exchange-degenerate ρ-f^0 trajectory. The residues of the s poles are polynomials in t, so that at a given mass there exists only a finite number of resonances with a maximum spin. The series of s poles add up to a function possessing Regge poles in the t channel—a phenomenon known as duality. More precisely,

$$\{s \text{ poles at } \alpha_s = 1, 2, \cdots\} \Rightarrow \{J \text{ poles at } J = \alpha_t, \alpha_t - 1, \cdots\}. \tag{2}$$

Unitarity is not satisfied, because there are no branch cuts in s. The obvious modification of replacing α_s and α_t by dispersion integrals fails, for the residues of the s poles would not remain polynomials in t, and consequently there would exist resonances of all spins (up to ∞) at the same mass.

Our generalization is motivated by the view[3] that a pole term in the Veneziano model is the Born approximation to some true propagator. By analogy with the Lehmann representation in field theory, we take

$$A(s;t) = \int_{x_0}^{\infty} dx \int_{x_0}^{\infty} dy\, \rho(x,y;s,t) V_0(\alpha_s - x, \alpha_t - y), \tag{3}$$

where s and t are complex numbers, and ρ is symmetric in x, y and in s, t. In general ρ must depend on s and t, and have branch points in s and t, in order that the Mandelstam double-spectral functions be different from zero; but the s, t dependences will be neglected in first approximation. For convergence of the integrals in (3), ρ should vanish faster than any power of x as $x \to \infty$. Owing to the x integration there occurs a branch cut in s in each and every pole term of V_0, with respective branch points $s_n = (n+\frac{1}{2}+x_0)/\alpha'$ $(n = 0, 1, 2, \cdots)$. These correspond to the elastic threshold $(n=0)$, and production thresholds $(n>0)$ for $\pi\pi \to AB$, where A and B are particles on the ρ-f^0 trajectory and daughter trajectories. Requiring the elastic threshold to be $s_0 = 0$ (in the approximation of vanishing pion mass), we have

$$x_0 = -\tfrac{1}{2}. \tag{4}$$

Since a branch cut is a continuous distribution of poles, by duality the infinite family of s cuts generate an infinite family of Regge cuts in the t channel. From the correspondence (2), we see that their respective branch points are at $J = \alpha_t + \frac{1}{2}$, $\alpha_t + \frac{3}{2}$, $\cdots$. The leading branch point is

$$\alpha_c = 1 + \alpha' t, \tag{5}$$

which will be our candidate for the Pomeranchuk sungularity.

Without restrictions on ρ, (3) is of course empty, for any function of s, t can be put in that form. It is our hope that ultimately the requirements of unitarity and analyticity will determine an equation for ρ–the dynamical equation. For the present, however, we merely examine the consequences of treating (3) as a small perturbation on the Veneziano model.

As a lowest-order perturbation on the Veneziano model we take ρ to be independent of s, t, thereby giving the amplitude absorptive parts, but no double-spectral function:

$$A(s,t) \approx V_1(s,t), \quad V_1(s,t) = \int_{-1/2}^{\infty} dx \int_{-1/2}^{\infty} dy\, f(x,y) V_0(\alpha_s - x, \alpha_t - y), \tag{6}$$

which reduces to the Veneziano model if $f(x,y) = \delta(x)\delta(y)$. We take $f(x,y)$ to be a continuous real function symmetric in x, y, with no real poles, but strongly peaked at $x=0$ and at $y=0$, with width γ. The discontinuity across the unitarity s cut will then go through maxima near the positions of the unperturbed poles. For convenience (though not necessity) we assume that $f(x,y)$ has complex conjugate poles at $x = \pm i\gamma$ and $y = \pm i\gamma$. Through a pinching of the contour of x integration[4] in (6), complex s poles will then occur near the real s axis on nonphysical Riemann sheets, and may be identified with unstable particles.

We state without proof some consequences of (6). The coefficients of the s-pole terms continue to be polynomials in t. If $f(x,y)$ is not of a factorized form $g(x)g(y)$, then the unperturbed poles are generally split, the nth pole splitting into $n+1$. If $f(x,y) = g(x)g(y)$, however, the degeneracies are not removed. (By what we are doing, of course, we cannot determine the true degeneracies of the unperturbed poles.[5]) The perturbed poles are all at a distance γ from the real s axis, to lowest order in γ. A resonance of mass M will have total decay width $\gamma/M\alpha'$. Partial unitarization consists of requiring $\gamma/M\alpha'$ to be greater than the $\pi\pi$ partial width calculated from pole residues, but we shall not discuss this here. By duality, what takes place in the s plane finds complete reflection in the J plane of the t channel. Namely, all unperturbed Regge poles recede into nonphysical Riemann sheets, and the nth daughter splits into n. The Regge cut (5), rather than the Regge poles, controls the asymptotic behavior, which we now examine more closely.

As $s \to \infty$, we have

$$V_1(s,t) \to -\beta \int_{-1/2}^{\alpha_s} dy\, F(y) \Gamma(1 - \alpha_t + y)(-\alpha' s)^{\alpha_t - y}, \tag{7}$$

where

$$F(y) = \int_{-1/2}^{\infty} dx\, f(x,y). \tag{8}$$

Note that the upper limit of the integral in (7) is cut off at α_s, where the integrand becomes a poor approximation. The rest of the integral is neglected because it is of order $\exp(-\alpha_s)$. For illustration we take

$$F(y) = (\gamma/\pi) e^{-y} (1+2y)^{\lambda} (y^2 + \gamma^2)^{-1} \quad (\gamma > 0,\ \lambda > 0). \tag{9}$$

As $\gamma \to 0$, $F(y) \to \delta(y)$, and (7) converges to the Veneziano model. The convergence is not uniform in s, making diffraction scattering possible. For sufficiently small γ, the integrand of (7) exhibits two sharp peaks, one at $y=0$ and the other at the end point $y = -\frac{1}{2}$. The former represents the contribution from the ρ-f^0 Regge trajectory, while the latter comes from the branch point (5) deduced earlier. We need only keep contributions to (7) coming from the neighborhood of these two peaks. Thus to first order in γ we find

$$V_1(s,t) \to -\beta \Gamma(1 - \alpha_t)(-\alpha' s)^{\alpha_t} - \beta\gamma \int_{\alpha_c - \epsilon}^{\alpha_c} dl\, D(l,t)(-\alpha' s)^{l}, \tag{10}$$

where ϵ is a suitable number, and

$$D(l,t)=\pi^{-1}2^{\lambda+2}e^{1/2}(\alpha_c-l)^{\lambda}\Gamma(1-l). \tag{11}$$

Because of the assumed smallness of γ, both terms in (10) are important at moderately high energies, exhibiting the combined effect of a Regge pole and a Regge cut. At extremely high energies the cut dominates:

$$V_1(s,t)\rightarrow-\beta\gamma e^{1/2}\pi^{-1}2^{\lambda+2}\Gamma(\lambda+1)\Gamma(1-\alpha_c)(-\alpha's)^{\alpha_c}(\ln\alpha's)^{-\lambda-1}. \tag{12}$$

This is valid outside of a shrinking neighborhood of $t=0$, of extension $\sim(\alpha's)^{-1}$. We expect important modifications of (12) to occur when the approximation (6) is improved to take into account more fully the requirements of unitarity and isospin crossing symmetry. Detailed isospin considerations, in the manner of Ref. 2, show that in the approximation (6) the branch point α_c occurs in the $I=0$ and the $I=1$ states in the t channel, but not in the $I=2$ state. The reason for its occurrence in the $I=1$ state is the rigid structure of the Veneziano model with respect to crossing, namely, all isospin amplitudes are determined by one function $V_0(\alpha_s,\alpha_t)$.[2] This suggests that a higher approximation would alter this crossing property.

A systematic scheme of approximation would involve a method to determine $\rho(x,y;s,t)$ by unitarity. In the absence of this, we could only speculate. In this spirit it may be instructive to give an example of a correction term that removes α_c from the $I=1$ state. Consider

$$A(s,t)\approx V_2(s,t),\quad V_2(s,t)=V_1(s,t)+\int_{-1/2}^{\infty}dx\int_{-1/2}^{\infty}dy\,g(x,y)W(d_s-x,d_t-y), \tag{13}$$

where $g(x,y)=g(y,x)$, and $W(x,y)$ has the properties

$$W(x,y)=W(y,x),\quad W(x,y)\underset{x\to\infty}{\longrightarrow}-\beta\Gamma(1-y)x^y\quad(y<1),\quad W(x,-x)\underset{x\to\infty}{\longrightarrow}O(e^{-x}). \tag{14}$$

As an example just to show that such a function exists, take

$$W(x,y)=U(x,y)+U(y,x),$$

where

$$U(x,y)=x\Gamma(1-y)[x^{1-y}+y^{y^2}]^{-1}.$$

In correspondence with (8) we choose

$$G(y)=\int_{-1/2}^{\infty}dx\,g(x,y)=(\gamma/\pi)e^{-y}. \tag{15}$$

The second term in (13) contains no resonances, and may be looked upon as a correction to the resonance tails in V_1. Since it is of order γ, it will not affect the Regge pole contributions in V_1, but will change the cut contribution. It is straightforward to verify that with (13) the branch point α_c occurs only in the $I=0$ state in the t channel. To first order in γ we then have the results

$$A(\pi^+\pi^-)\approx V_2(s,t)\underset{s\to\infty}{\longrightarrow}A_{\text{pole}}\exp(-i\pi\alpha_t)+A_{\text{cut}},\quad A(\pi^+\pi^+)\approx V_2(u,t)\underset{s\to\infty}{\longrightarrow}A_{\text{pole}}+A_{\text{cut}}, \tag{16}$$

where

$$A_{\text{pole}}=-\beta\Gamma(1-\alpha_t)(\alpha's)^{\alpha_t},\quad A_{\text{cut}}=-\gamma\beta\int_{\alpha_c-\epsilon}^{\alpha_c}dl\,D(l,t)[1+\exp(-i\pi l)](\alpha's)^l, \tag{17}$$

where ϵ and $D(l,t)$ are the same as in (10). At extremely high energies at $t=0$,

$$A(\pi^+\pi^-)\approx A(\pi^+\pi^+)\approx A_{\text{cut}}\approx i\beta\gamma e^{1/2}2^{\lambda+2}\Gamma(\lambda+1)(\alpha's)(\ln\alpha's)^{-\lambda-1}. \tag{18}$$

Thus the branch point α_c is a possible candidate for the Pomeranchuk singularity. In our picture it is indeed made up of contributions from resonance tails, as conjectured by Harari.[6]

Finally we comment on the pion mass. One of the most attractive features of Lovelace's treatment of the $\pi\pi$ problem[2] is the occurence of a zero of the scattering amplitude at $1-\alpha_s-\alpha_t=0$, which coincides with that required by the Adler self-consistency condition,[7] provided $\alpha_0+\alpha'm_\pi^2=\frac{1}{2}$, where m_π is the pion mass. Instead of regarding this condition as determining α_0 in terms of the experimental value of m_π, we view the Veneziano model as a zero-order approximation with $\alpha_0=\frac{1}{2}$, $m_\pi=0$. Then in principle we can calculate the pion mass by locating the position of the Adler zero in the unitarized amplitude (3). To lowest order in m_π this gives

$$\alpha' m_\pi^2 = \tfrac{1}{2}\langle x+y\rangle, \tag{19}$$

where the angular brackets denote averaging with respect to the weighting function $\rho(x,y;0,0)B(\frac{1}{2}+x, \frac{1}{2}+y)$, where B is the β function. Needless to say such a calculation cannot be now made because it depends on details of the function ρ that are yet unknown. A crude estimate, however, may be obtained as follows. In the form (1) every pole term separately vanishes at the Adler point. We can estimate the order of magnitude of the shift of the zero by estimating the shift of the zero in the real part of the first pole term. To do this we simply assume that the unitarity corrections to the first pole term in (1) roughly consist of replacing α_s by $\alpha_s + i\gamma$, where

$$\gamma = \alpha' m_\rho \Gamma_\rho, \tag{20}$$

m_ρ and Γ_ρ being the mass and width of the ρ meson, respectively. This procedure gives

$$m_\pi \approx (\alpha')^{1/2} m_\rho \Gamma_\rho = 95 \text{ MeV}. \tag{21}$$

A finite pion mass will change the threshold (4) and depress the intercept of α_c by a number of order m_π^2. Thus in a completely self-consistent scheme, we expect the Pomeranchuk singularity and the pion mass to be coupled together.

I thank S. Fubini, Huan Li, and G. Veneziano for interesting discussions.

*Work supported in part through funds provided by the U. S. Atomic Energy Commission under Contract No. AT(30-1)-2098.

[1]G. Veneziano, Nuovo Cimento, 57A, 190 (1968).

[2]C. Lovelace, Phys. Letters, 28B, 265 (1968).

[3]I thank S. Fubini for expressing and emphasizing this view.

[4]For this remark I thank Mr. Frank Paige.

[5]There is indication that the nth unperturbed pole is $P(n)$-fold degenerate, where $P(n)$ is the partitio numerorum of n. See S. Fubini and G. Veneziano, to be published.

[6]H. Harari, Phys. Rev. Letters, 20, 1395 (1968).

[7]S. Adler, Phys. Rev., 137, B1022 (1965).