matrix element in Eq. (4) has no subtractions in p^2 or k^2. In any case, unless we impose some symmetry constraint on $F_\pi(t)$ such as proposed by Suura,[6] $F_\pi(t)$ will now depend on many unknown inelastic amplitudes involving the heavy π and A mesons.

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UNSTABLE PARTICLES, TWO-BODY INELASTIC UNITARITY, AND VENEZIANO'S MODEL*

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We propose an integral over the Veneziano amplitude which introduces singularities of two-body inelastic unitarity in the amplitude. The model provides a framework for a unified treatment of quantization conditions, the stability of particles, the Pomeranchuk singularity, and decay of particles by pion emission.

Recently a model for the strong amplitude has been suggested by Veneziano[1] which has a number of desirable properties but is in disagreement with unitarity. In this note we propose a generalization which partially remedies this difficulty. We take the view that the Veneziano amplitude is in some sense a zeroth approximation to the correct amplitude, and its success suggests that the corrections are small. We write the Veneziano amplitude as[2]

$$A_V(s,t)=\frac{\Gamma(J^{\min}-\alpha(s))\Gamma(J^{\min}-\alpha(t))}{\Gamma(2J^{\min}+l-\alpha(s)-\alpha(t))}, \tag{1}$$

where $J^{\min}$ is the lowest physical angular momentum on α, and l is an appropriate integer ≤ 0 and $J^{\min}\geq |l|$. We take $\alpha(y)=a+by$. The fundamental Ansatz of our model is

$$\begin{aligned}A(s,t)=\int_{-\infty}^{L}da\,\rho(a,a^0,L,\epsilon,\cdots)\Bigg\{&\frac{\Gamma(J^{\min}-a-bs)\Gamma(J^{\min}-a-bt)}{\Gamma(l+2J^{\min}-2a-bs-bt)}\\&-\sum_{n=0}^{k-1}\frac{(-1)^n}{n!}\left[\frac{\Gamma(J^{\min}-a-bt)}{\Gamma(l+J^{\min}-a-bt-n)(J^{\min}-a-bs+n)}+\frac{\Gamma(J^{\min}-a-bs)}{\Gamma(l+J^{\min}-a-bs-n)(J^{\min}-a-bt+n)}\right]\Bigg\}\\&+\sum_{n=0}^{k-1}\frac{(-1)^n}{n!}\left[\frac{\Gamma(J^{\min}-a^0-bt)}{\Gamma(l+J^{\min}-a^0-bt-n)(J^{\min}-a^0-bs+n)}+\frac{\Gamma(J^{\min}-a^0-bs)}{\Gamma(l+J^{\min}-a^0-bs-n)(J^{\min}-a^0-bt+n)}\right],\end{aligned} \tag{2}$$

which amounts to "smearing" with a weight ρ all but k poles of the Veneziano amplitude about the intercept of their trajectory. Clearly the above form is not the most general one[3] but has the virtue of being quite simple and yet possessing a number of very desirable features. If the deviations from the Veneziano form of Eq. (1) are small, our generalization should be reasonably good. Note that one can write partial-fraction (pole) expansions for the Veneziano form in the integrand of Eq. (2) and see that Eq. (2) has the form of a dispersion integral in either s or t with ρ being a weight function. Although the form of

ρ is a dynamical question, some general results can be obtained without knowledge of its exact form. For example, our results concerning trajectory intercepts and stability of particles depend only on the positions of the branch points which are introduced by our representation. We shall choose ρ to be strongly peaked about a^0, to tend to $\delta(a-a^0)$ as $\epsilon \to 0$ [thus $A(s,t) \to A_v(s,t)$ in this limit], and to be regular in a neighborhood of the negative real a axis. The simplest example of such a function is

$$\rho(a, a^0, L, \epsilon) \propto \frac{\epsilon f(a, a^0, L, \epsilon)}{(a-a^0)^2 + \epsilon^2}. \quad (3)$$

For this particular ρ we will derive properties of the scattering amplitude and thus show that the general representation can provide a good description of some processes. We shall take $L<1$ in order to preserve the Froissart limit, and $a^0 \leq L$ to maintain the $\epsilon \to 0$ limit. Choosing f to have no poles in a and such that the integral in Eq. (2) converges, we obtain the following results:

(i) The amplitude has branch points in s at

$$s_t^{\,n} = (1/b)(J^{\min} - L + n + k), \quad n = 0, 1, 2, \cdots, \quad (4)$$

and similarly for t. The nature of the cuts will depend on f. It is natural to associate these cuts with the elastic and two-body inelastic s and t cuts.

(ii) The amplitude has poles at

$$s_p^{\,n} = (1/b)[J^{\min} - a^0 + n \pm i\epsilon\theta(n-k)], \quad n = 0, 1, 2, \cdots. \quad (5)$$

If $n \leq k$, these poles are on the real axis and appear on the physical sheet, but if $n > k$ they are off the real axis at a distance $\pm i\epsilon$ from it and on an unphysical sheet. The residues of these poles will be polynomials in t of order $n-l$. It is reasonable to associate these poles with stable and unstable particles.

(iii) As $|s| \to \infty$ the $\pm i\epsilon$ poles on the unphysical sheet contribute Regge-like behavior to the physical amplitude, of the form

$$A \sim f(t)(-s)^{m + a^0 + bt \pm i\epsilon} \quad (6)$$

with $m = -l - J^{\min}$. This is obtained by rotating contours in the a plane. To order ϵ there is also a contribution from the background integral which depends on the choice of f. To order ϵ there is also a contribution from fixed pole terms coming from terms in the sums over n in Eq. (2) which nearly cancel.

We begin by examining results of the model which are independent of the specific choice of ρ, depending only on positions of singularities. Since the model yields an infinite number of thresholds, it is necessary to see if their positions can be made consistent with the particle spectrum of the model. A physically attractive interpretation of our spectrum of thresholds is the stepwise decay of resonances through emission of massless pions. This interpretation appears to be confirmed by our investigation of the connection with partial conservation of axial-vector current.

Equation (5) describes a set of poles and Eq. (4) a set of thresholds of the amplitude. If we assume that the external particles also lie on a trajectory and thus satisfy a condition analogous to Eq. (5), we can demand consistency between external poles and thresholds. This consistency can be achieved only if one of the external particles is massless. Labeling the massless particle P, the other external trajectory A, and the internal trajectory X, the obvious interpretation of our model is that we are incorporating the decay of X into A through emission of a massless pion. To order ϵ we have, matching poles and thresholds,

$$\frac{1}{b_A}(J_A^{\min} - a_A^{\,0} + n) \approx \frac{1}{b_X}(J_X^{\min} - L_X + n + k_X), \quad n = 0, 1, 2, \cdots, \quad (7)$$

from which it follows that $b_A = b_X$ (universality of slopes) and

$$J_A^{\min} - a_A^{\,0} \approx J_X^{\min} - L_X + k_X. \quad (8)$$

We can now consider an amplitude with X and A exchanged. Although this amplitude will be quite different from the previous one, it will again be true that in our model

$$J_X^{\min} - a_X^{\,0} \approx J_A^{\min} - L_A + k_A. \quad (9)$$

The fact that particle P is a massless pion suggests that we take into account partial conservation of axial-vector current in the fashion suggested by Lovelace,[4] Ademollo, Veneziano, and Weinberg,[5] and Coon.[6] This is nontrivial when A and X are trajectories of opposite normality. (Our parity considerations and notation are the same as in Ref. 5.) Our amplitude will vanish at the required spot to order ϵ. Now Coon has pointed out[6] that the intercepts of trajectories with opposite normality can differ by $\frac{1}{2}$ ($a_X^{\,0} = a_A^{\,0} + \frac{1}{2}$) only if $J_X^{\min} - J_A^{\min} \geq 1$. These conditions together with Eq. (9) imply that $k_A \geq \frac{1}{2} + L_A - a_A{}^0$ or, as $L_A \geq a_A{}^0$, it must be that $k_A \geq \frac{1}{2}$. Thus there must be at least one stable particle on trajectory A. Therefore it is necessary to include terms outside the smearing integral in Eq. (2). By admitting at most one stable particle we can avoid undesirable power behavior from these terms. This is because the spin j of a stable particle is related to n by $j = n + J^{\min}$. The asymptotic behavior of the nth term is $(-s)^{j-\Delta}$ where $\Delta = l + J^{\min}$ is the usual helicity-flip index. Since $J^{\min} \geq 0$, only the $n=0$ term is compatible with the Froissart bound ($j<1$). Thus, there can be at most one stable particle on any trajectory.

Furthermore, as k_A is an integer,

$$L_A = a_A{}^0 + \tfrac{1}{2} + m, \quad m = 0, 1, \cdots, \tag{10}$$

subject to $L_A \leq 1$, or $a_A{}^0 \leq \frac{1}{2}$. Similarly, Eq. (8) implies that

$$L_X = a_X{}^0 + \tfrac{1}{2} + n, \quad n = 0, 1, \cdots, \tag{11}$$

with $a_X{}^0 \leq \frac{1}{2}$. If X, the higher trajectory, has intercept a^0 greater than zero, then $m = n = 0$. These considerations indicate that no trajectory can have intercept higher than $\frac{1}{2}$. All the above conditions seem to be approximately satisfied by the usual pairs of trajectories with opposite normality: ρf-πA_1; Δ-N, K^*-K, and $Y_1{}^*$-Σ. In the case of ρf-πA_1 in the $\pi\pi$ system, Eq. (4) and the fact that $J_{\rho f}{}^{\min} = 1$ indicate that $L_{\rho\text{-}f} = 1$, so that $a_{\rho f}{}^0 = \frac{1}{2}$, and the agreement is particularly good.

To illustrate other properties of the model, we shall choose ρ to be

$$\rho(a, a^0, L, \epsilon) = \frac{\epsilon/\pi (a-L)^{1/2}}{(a-a^0)^2 + \epsilon^2}. \tag{12}$$

It is then easy to see that the branches introduced at the two-body inelastic thresholds are square-root branches, and the resonance pole associated with a particular cut is found only on those sheets reached by circling the corresponding threshold. The poles appear in complex conjugate pairs, thus preserving the real analyticity of the amplitude. To study the asymptotic behavior of the amplitude, it is convenient to change the integral in Eq. (2) to a contour integral, with the contour running around the cut of ρ to $-\infty$, on the arc at infinity, and closing on the line $\mathrm{Re}(a) = L$. Then, if $|s| \to \infty$ with $\mathrm{Re}s < 0$, $\mathrm{Re}(t) < 0$, the only poles encompassed by the contour are those of ρ, yielding

$$A_p(s,t) \propto (a_L - a^0 - i\epsilon)^{1/2} B(1-s-a^0-i\epsilon, 1-t-a^0-i\epsilon) + (a_L - a_0 + i\epsilon)^{1/2} B(1-s-a^0+i\epsilon, 1-t-a^0+i\epsilon), \tag{13}$$

where B is the Euler β function.

The pole terms exhibit the usual Regge power behavior times an oscillatory component of the form $\cos(\epsilon \ln bs)$ which is unimportant for $\epsilon \ln bs \ll 1$. The integral along the line $\mathrm{Re}(a) = L$ contributes a background term bounded by $\epsilon h(t)|s|^{L+t}/(\ln s)^{3/2}$ as s grows large. The presence of the logarithm would seem to indicate the presence of J-plane cuts. Specifically, it is very tempting to associate this term with the Pomeranchuk singularity, in which case L must be very close to 1. This association has the very attractive feature of connecting the Pomeranchon with a "background" contribution and not with direct-channel resonances, as conjectured by Harari.[7] It also fixes, through Eq. (11), the intercept of the ρ to be very close to $\frac{1}{2}$, which is then consistent with a zero-mass pion. The agreement is not as good for the N-Δ or K^*-K case.

The difficulty with the above interpretation is that the Pomeranchuk singularity thus constructed will not have the correct isospin properties. For instance, in $\pi\pi$ scattering, if we use amplitudes of well-defined isospin with crossing and Bose symmetry and without $I=2$ resonances,[8] the background will contribute to both the $I=0$ and $I=1$ amplitudes. This difficulty can probably be resolved by adding appropriate terms to our model, but in the absence of a dynamical scheme, such a procedure must be ad hoc. If we now let $|s| \to \infty$ with $\mathrm{Re}(s) > 0$, poles of the integrand in the right half plane migrate to the left half plane and into the contour, after pinching at L and giving rise to the threshold branch points. These poles are connected with the resonance poles in s for $s > 0$ and their residue in the physical sheet vanishes. The asymptotic considerations for $\mathrm{Re}(s) < 0$ remain valid.

The model we have described has the analytical structure suggested by two-body unitarity with an infinite number of channels, and with the channels given by all possible "$\pi \cdot A$" resonances. It is clearly, however, not yet unitarity. It preserves the properties and simplicity of the Veneziano model and it suggests that certain particles must be stable for consistency. It also implies a possible role for the Pomeranchuk singularity which is strikingly different from that of the other singularities in l.

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[2]For simplicity we are taking the t and u channels as identical.

[3]An infinity of different forms comes immediately to mind. Notice, for instance, that in order to satisfy unitarity exactly, ρ should depend on s and t to introduce double spectral functions. Also in nonsymmetric cases one could expect to smear separately in s and t, etc. It is hoped that as $\epsilon \to 0$, our form retains the important features.

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UNITARIZATION OF THE VENEZIANO MODEL, POMERANCHUK SINGULARITY, AND THE PION MASS*

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We suggest a general framework for unitarizing the Veneziano model for $\pi\pi$ scattering, and deduce some immediate consequences. By duality the unitarity s cut generates a moving Regge cut with branch-point trajectory $1+\alpha' t$ in the approximation $m_\pi = 0$, with α' the slope of the ρ-f^0 trajectory. In principle m_π may be calculated from the shift of the Adler zero. A crude estimate yields $m_\pi \approx (\alpha')^{1/2} m_\rho \Gamma_\rho = 95$ MeV.

We suggest a general framework in which the unitarization of the Veneziano model[1] might be attempted, and describe some immediate qualitative consequences. Consider $\pi^+\pi^-$ scattering in the approximation of vanishing pion mass. The scattering amplitude in the Veneziano model is given by[2]

$$A(s,t) \approx V_0(\alpha_s, \alpha_t),$$

$$V_0(\alpha_s, \alpha_t) = \frac{-\beta\Gamma(1-\alpha_s)\Gamma(1-\alpha_t)}{\Gamma(1-\alpha_s-\alpha_t)} = \beta(1-\alpha_s-\alpha_t)\sum_{n=0}^{\infty}\binom{\alpha_t+n-1}{n}\frac{1}{\alpha_s-n-1}, \tag{1}$$

where $\alpha_t = \frac{1}{2}+\alpha' t$ is the exchange-degenerate ρ-f^0 trajectory. The residues of the s poles are polynomials in t, so that at a given mass there exists only a finite number of resonances with a maximum spin. The series of s poles add up to a function possessing Regge poles in the t channel—a phenomenon known as duality. More precisely,

$$\{s \text{ poles at } \alpha_s = 1, 2, \cdots\} \Rightarrow \{J \text{ poles at } J=\alpha_t, \alpha_t-1, \cdots\}. \tag{2}$$

Unitarity is not satisfied, because there are no branch cuts in s. The obvious modification of replacing α_s and α_t by dispersion integrals fails, for the residues of the s poles would not remain polynomials in t, and consequently there would exist resonances of all spins (up to ∞) at the same mass.

Our generalization is motivated by the view[3] that a pole term in the Veneziano model is the Born approximation to some true propagator. By analogy with the Lehmann representation in field theory, we take

$$A(s;t) = \int_{x_0}^{\infty} dx \int_{x_0}^{\infty} dy\, \rho(x,y;s,t) V_0(\alpha_s-x, \alpha_t-y), \tag{3}$$

where s and t are complex numbers, and ρ is symmetric in x, y and in s, t. In general ρ must depend on s and t, and have branch points in s and t, in order that the Mandelstam double-spectral functions be different from zero; but the s, t dependences will be neglected in first approximation. For convergence of the integrals in (3), ρ should vanish faster than any power of x as $x \to \infty$. Owing to the x integration there occurs a branch cut in s in each and every pole term of V_0, with respective branch points $s_n = (n+\frac{1}{2}+x_0)/\alpha'$ $(n=0,1,2,\cdots)$. These correspond to the elastic threshold $(n=0)$, and production thresholds $(n>0)$ for $\pi\pi \to AB$, where A and B are particles on the ρ-f^0 trajectory and daughter trajectories. Requiring the elastic threshold to be $s_0=0$ (in the approximation of vanishing pion mass),