Contract No. AT(30-1)-2137, and Office of Naval Research Contract No. N0014-67-A-0151-0001.

[1]J. Bernstein, G. Feinberg, and T. D. Lee, Phys. Rev. 139, B1650 (1965).

[2]S. Barshay, Phys. Letters 17, 78 (1965).

[3]See, for instance, J. C. Keck and A. V. Tollestrup, Phys. Rev. 101, 360 (1956). More recent work has been done by R. Kose, W. Paul, K. Stockhorst, and K. H. Kissler, Z. Physik 202, 364 (1967); J. Buon, V. Gracco, J. Lefrançois, P. Lehmann, B. Merkel, and Ph. Roy, Phys. Letters 26B, 595 (1968); D. I. Sober, D. G. Cassel, A. J. Sadoff, K. W. Chen, and P. A. Crean, Phys. Rev. Letters 22, 430 (1969); C. A. Tatro, T. R. Palfrey, R. M. Whaley, and R. O. Haxby, Phys. Rev. 112, 932 (1958); E. A. Whalin, B. D. Schriever, and A. O. Hanson, Phys. Rev. 101, 377 (1956); Yu. A. Aleksandrov, N. B. Delone, L. I. Slovokhotov, G. A. Sokol, and L. N. Shtarkov, Zh. Eksperim. i Teor. Fiz. 33, 614 (1957) [translation: Soviet Phys.—JETP 6, 472 (1958)]; A. M. Smith, S. J. Hall, B. Mann, and D. T. Stewart, J. Phys. A: Phys. Soc. (London) Proc. 1, 553 (1968).

[4]S. Barshay, Phys. Rev. Letters 17, 49 (1966).

[5]We have studied this background process since it provides a test of charge independence. We have found that both the energy dependence of the total cross section and the angular distributions are in excellent agreement with existing data for $n+p\to d+\pi^0$ and $p+p\to d+\pi^+$. I. Hammerman, thesis, Princeton University, Palmer Physical Laboratory Report No. PURC-2137-9 (unpublished).

[6]Monte Carlo program based on technique of J. Butcher and H. Messell, Nucl. Phys. 20, 15 (1960). We have checked this calculation by using γ rays from the reaction $n+p\to d+\pi^0$.

[7]T. Devlin, P. Shepard, R. Mischke, and J. Solomon, private communication.

[8]A possible source of systematic error could arise if the neutron beam were polarized. Reciprocity requires that the angular distributions for Reactions (1) and (2) be the same if initial-state spins are averaged and final-state spins summed. We have measured the polarization of the incident neutrons in our beam and found it to be 0.04 ± 0.02 ($T_n\approx 600$ MeV), a value too small to yield significant asymmetries. Our detector is insensitive to the spin of either the deuteron or the photon. A parallel situation holds for the measurement of Reaction (2).

[9]B. L. Schrok, J.-F. Detoeuf, R. P. Haddock, J. Helland, M. J. Longo, K. K. Young, S. S. Wilson, D. Cheng, and V. Perez-Mendez, private communication; M. Longo, Bull. Am. Phys. Soc. 14, 598 (1969).

CONNECTION BETWEEN $F_\pi(t)$ AND THE AMPLITUDES FOR $\pi\pi\to\pi\pi$ AND $\pi\pi\to\pi A_1$ *

D. A. Geffen
School of Physics and Astronomy, University of Minnesota, Minneapolis, Minnesota 55455
(Received 10 September 1969)

The current-algebra connection between the pion electromagnetic form factor $F_\pi(t)$ and the amplitudes for $\pi\pi\to\pi\pi$ and $\pi\pi\to\pi A_1$ is examined by explicitly extrapolating off the pion mass shell the matrix element $\langle\pi|A_\mu|\pi\pi\rangle$ which is taken to be dominated by π and A_1 poles and possible subtractions. It is found that the connection is broken by the presence of an almost arbitrary subtraction function. In particular the $A\rho\pi$ interaction remains arbitrary as well as the form for $F_\pi(t)$. The results are applied briefly to the Veneziano model.

Several authors[1] have recently explored the connection between the amplitudes $\pi\pi\to\pi\pi$ and $\pi\pi\to\pi A_1$ that follows from the assumptions that the matrix element $\langle\pi\pi|A_\mu|\pi\rangle$ is dominated by π and A_1 poles and that the scattering amplitudes are given by the Veneziano model.[2] Oyanagi[3] has extended the analysis by extrapolating to zero the momentum of one of the pions in the matrix element and thereby obtaining a Veneziano-like expression for $F_\pi(t)$. One result of this work, and that of a subsequent analysis,[4] has been to propose a vanishing D-wave $A\rho\pi$ interaction, either to yield a more convergent $F_\pi(t)$ for large t or to eliminate satellites from the $\pi\pi\to\pi\pi$ and $\pi\pi\to\pi A_1$ scattering amplitudes. In all these treatments, however, the extrapolation of one pion off the mass shell of the amplitude $\langle\pi\pi|A_\mu|\pi\rangle$ has not been done in a consistent way.[5] We intend to summarize the results of such a consistent treatment in this Letter and show that the use of the Veneziano model for this example imposes no restriction on the D-wave $A\rho\pi$ interaction and that one is free to obtain almost any asymptotic behavior for $F_\pi(t)$.

Consider the amplitude for $\pi^+(p)+\pi^-(q)\to A_\mu^{\ +}(k)+\pi^-(q')$ with the π^+ extrapolated off the mass shell,

$$M_\mu=\tfrac{1}{2}\int dx\, e^{-ipx}\langle\pi^-(q')|T[\partial^\lambda A_\lambda^{\ +}(x),A_\mu^{\ -}(0)]|\pi^-(q)\rangle. \qquad (1)$$

Here $k+q'=p+q$ and $px=p^\lambda x_\lambda=p_0x_0-\vec{p}\cdot\vec{x}$. We take the conventional current-algebra commutation rules,

$$[A_0^+(x), A_\mu^-(0)]\delta(x_0) = 2\delta(x)V_\mu^3 + \text{S.T.}, \quad [A_0^-(x), \partial^\lambda A_\lambda^+(0)]\delta(x_0) = 2i\delta(x)\sigma, \tag{2}$$

and derive two current-algebra conditions on M_μ [assuming, as usual, that the Schwinger terms (S.T.) do not contribute],

$$\lim_{p\to 0} M_\mu = F_\pi(t)(q+q')_\mu, \tag{3}$$

$$k^\mu M_\mu = \tfrac{1}{2}i\int dx\, e^{-ipx}\langle\pi^-|T[\partial^\lambda A_\lambda^+(x), \partial^\mu A_\mu^-(0)]|\pi^-\rangle - \Sigma(t) = [m_\pi^4 f_\pi^2/(m_\pi^2-p^2)(m_\pi^2-k^2)]A(s,t) - \Sigma(t). \tag{4}$$

$A(s,t)$ is the $\pi^+\pi^-$ elastic-scattering amplitude and $\Sigma(t) = \langle\pi^-(q')|\sigma|\pi^-(q)\rangle$. In writing $k^\mu M_\mu$ in this way, we are making the additional assumption that the $\pi\pi$ scattering amplitude can be extrapolated off the mass shell in a maximally "smooth" way. We cannot make such an assumption for matrix elements of $\partial^\lambda A_\lambda$ involving the A_1 meson. Next, let us assume the following form for M:

$$M_\mu = [m_\pi^2 f_\pi/(m_\pi^2-p^2)]\{[f_\pi/(m_\pi^2-k^2)]A(s,t)k_\mu + [F_A/(m_A^2-k^2)][B_0k_\mu + B_iq_\mu + B_2(s,t)(p+q')_\mu] + C_0k_\mu + C_iq_\mu + C_2(p+q')_\mu\} + [F_A/(m_A^2-k^2)][D_0k_\mu + D_1q_\mu + D_2(p+q')_\mu] + [f_\pi/(m_\pi^2-k^2)]E(s,t)k_\mu. \tag{5}$$

The $B_i(s,t)$ are the $\pi^+\pi^-\to A_1^+\pi^-$ amplitudes with

$$m_A^2B_0 + (s-m_\pi^2)B_+ + (t-m_\pi^2)B_- = 0, \tag{6}$$

as required by the fact that $\partial^\mu A_\mu^-$ does not couple to 1^+ mesons. We have introduced the notation $B_\pm = \frac{1}{2}(B_1 \pm B_2)$, which will be applied in the future to the amplitudes $C_{1,2}$ and $D_{1,2}$ as well. The $C_i(s,t)$ are subtraction terms in k^2 for the amplitude $\langle\pi^-|A_\mu^-|\pi^+\pi^-\rangle$ and have been considered previously by several authors.[1,4,5] The $D_i(s,t)$, however, are subtractions in p^2 for the matrix element $\langle\pi^-A_1^+|\partial^\lambda A_\lambda^+|\pi^-\rangle$ and violate the notion of π-pole dominance of the divergence of the axial-vector current. We will show, shortly, that in general the $D_i \neq 0$ if we are to satisfy Eq. (4). This should not be surprising since it has been well known that simple π-pole dominance of $\partial^\lambda A_\lambda$ cannot be maintained in such amplitudes. For example, in effective-Lagrangian models with $\partial^\lambda A_\lambda = m_\pi^2 f_\pi\varphi$ one can easily generate chiral-invariant interactions which produce such subtraction terms. On the other hand, $E(s,t)$ in Eq. (5) represents a subtraction for $\langle\pi^-\pi^+|\partial^\lambda A_\lambda^+|\pi^-\rangle$ and is not consistent with Eq. (4). Consequently we will find $E=0$. An additional subtraction term in M_μ corresponding to a subtraction in both p^2 and k^2 has been omitted [it must appear in the form $(q=p=q')_\mu F_+(s,t)$ unless we add additional polynomials in p^2 and k^2 to M_μ]. While such a term cannot be ruled out, we will see that enough arbitrariness is present without including it in M_μ. It should be noted that the subtraction term cannot be taken too literally as such terms may, in fact, vanish as p^2 or $k^2\to\infty$ if they arise from high-mass 0^- and 1^+ states. Since $k^2=t$ when $p=0$, this raises serious doubts about the reliability of Eq. (3) in the large-t limit. In fact, we will show that the freedom in the choice of subtraction terms enables us to give $F_\pi(t)$ many different asymptotic forms.

It is straightforward algebra to use this expression for M_μ to apply Eq. (4). With $B_\pm = \frac{1}{2}(B_1 \pm B_2)$, $C_\pm = \frac{1}{2}(C_1 + C_2)$, etc., we find

$$C_0(s,t) = D_-(s,t) = E(s,t) = 0, \tag{7a}$$

$$f_\pi A(s,t) = (s-m_\pi^2)[(F_A/m_A^2)B_+ + C_+] + (t-m_\pi^2)[(F_A/m_A^2)B_- + C_-], \tag{7b}$$

$$m_\pi^2 f_\pi B_- + m_A^2 D_0 + (s-m_\pi^2)D_+ = 0, \tag{7c}$$

$$D_0 = \Sigma(t)/F_A + (m_\pi^2 f_\pi/F_A)C_-. \tag{7d}$$

Crossing symmetry $(p \leftrightarrow -q')$ when $\pi^+(p)$ is on the mass shell yields simple crossing properties for the terms in M_μ proportional to $(m_\pi^2-p^2)^{-1}$:

$$A(t,s) = A(s,t); \quad B_+(t,s) = B_-(s,t); \quad C_+(t,s) = C_-(s,t). \tag{8}$$

Equation (7c) tells us that all the D_i cannot vanish and Eqs. (7b) and (7d), when combined with the crossing properties of $C_\pm$, require $D_0 \neq 0$ if we are to avoid fixed poles in either of the physical scattering amplitudes $A(s,t)$ and $B_\pm(s,t)$. To see this take $D_0=0$ so that $C_- \propto \Sigma(t)$ and $C_+ \propto \Sigma(s)$. Then either $A(s,t)$ contains a term $(s-m_\pi^2)\Sigma(s) + (t-m_\pi^2)\Sigma(t)$ (i.e., a fixed pole) or the $B_\pm$ have fixed poles to cancel this contribution from $C_\pm$. The limit $p\to 0$ of M_μ, when compared with Eq. (3), leads to the two relations

$$A(m_\pi^2,t)=[(m_\pi^2-t)/m_\pi^2 f_\pi^2]\Sigma(t), \tag{9}$$

$$F_\pi(t)=[m_A^2m_\pi^2/(m_A^2-t)(m_\pi^2-t)]f_\pi^2(\partial A/\partial s)_{s=m_\pi^2}-[F_Af_\pi/(m_A^2-t)(m_\pi^2-t)]tB_+(m_\pi^2,t)$$
$$+[1-m_A^2m_\pi^2/(m_A^2-t)(m_\pi^2-t)]f_\pi C_+(m_\pi^2,t). \tag{10}$$

Equation (9) is a well-known result and can be obtained directly by taking $\lim_{k\to 0}$ of $k^\mu M_\mu$ in Eq. (4). To obtain Eq. (10) for $F_\pi(t)$ we have made use of Eqs. (7) to eliminate $D_+(m_\pi^2,t)$ and exhibit explicitly the connection between the normalization of F_π and the Adler-Weisberger relation for A:

$$F_\pi(0)=1=f_\pi^2[\partial A(m_\pi^2,0)/\partial s].$$

Requiring $F_\pi(0)$ to be unity, therefore, yields no new information. So far we have made no assumptions about the form for $A(s,t)$, etc. Before taking Veneziano models for $A(s,t)$ and $B_\pm(s,t)$, let us take Eq. (10) seriously for large t and assume Regge behavior for $A(s,t)$ and $B_+(s,t)$, $A\sim t^{\alpha_P(s)}$, $B_+\sim t^{\alpha_P(s)}$, where $\alpha_P(s)$ is the Pomeranchuk trajectory. Consequently, for large t,

$$F_\pi(t)\sim\beta_+t^{\alpha_P(m_\pi^2)-1}+f_\pi C_+(m_\pi^2,t).$$

While $C_+(s,t)$ is related to the physical amplitudes A and $B_\pm$ by Eq. (7b), it is not completely determined by it. By virtue of the crossing relations in Eq. (8), only the symmetric part of the function $(s-m_\pi^2)C_+(s,t)$ contributes to $A(s,t)$. Consequently, we can add to $(s-m_\pi^2)C_+$ any odd function of s,t without affecting A. For example, take $C_+=(t-m_\pi^2)(s-t)f(s,t)$, with $f(s,t)=f(t,s)$. Since this part of C_+ does not contribute to a physical hadronic scattering amplitude (in principle the $C_\pm$ can be "measured" in neutrino pion production on pions), we cannot determine $F_\pi(t)$ from known physical amplitudes. Furthermore, since $\alpha_P(m_\pi^2)\cong 1$, $C_+(m_\pi^2,t)$ cannot vanish without leaving $F_\pi(t)\sim$ constant for large t. We conclude, therefore, that it is unreasonable to link the large-t behavior of $F_\pi(t)$ with the current-algebra approach at least until we learn more about the large-k^2 behavior of matrix elements of A_μ.

In the final paragraphs, we will make a few observations on the Veneziano model as it applies to $F_\pi(t)$. First it should be clear from our discussion above that representing $A(s,t)$ and $B_\pm(s,t)$ with the now conventional Veneziano forms,[2]

$$A(s,t)=\eta B_1^{11}(s,t),\quad B_+(s,t)=B_-(t,s)=\gamma(1+\xi\alpha_t)B_2^{11}(s,t) \tag{11}$$

[with $B_n^{km}(s,t)\equiv\Gamma(k-\alpha_s)\Gamma(m-\alpha_t)/\Gamma(n-\alpha_s-\alpha_t)$ and $\alpha_s=\frac{1}{2}+(s-m_\pi^2)/2(m_\rho^2-m_\pi^2)$, the ρ trajectory], cannot be applied to $F_\pi(t)$ for large t. On the other hand, it is possible that Eq. (11) can be a valid description (in some average sense) of A and $B_\pm$ in the intermediate range for $|t|\lesssim m_A^2$. Use of Eq. (11) in Eq. (10), however, does not impose any restrictions on A or $B_\pm$ because of the arbitrary nature of C_+. In particular we can obtain any number of expressions for $F_\pi(t)$ for arbitrary values of ξ, or, equivalently, for any ratio of D- to S-wave $A\rho\pi$ interactions. Of course, if the expressions for $B_\pm(s,t)$ in (11) are inserted in Eq. (7b) for A, then a "satellite" term $[\sim(s-m_\pi^2)(t-m_\pi^2)B_2^{11}(s,t)]$ is generated. While we really see nothing wrong with such a term, it can be eliminated if C_+ contains a term $-(\gamma\xi bF_A/m_A^2)(t-m_\pi^2)B_2^{11}(s,t)$ to cancel it. As an example, take for $C_\pm$,

$$C_+(s,t)=C_-(t,s)=-(\gamma bF_A/m_A^2)(t-m_\pi^2)[\xi B_2^{11}(s,t)-(\alpha_s-\alpha_t)\sum_{n=3}^{\infty}a_nB_n^{11}(s,t)]$$
$$+C(\gamma F_A/m_A^2)(1+\tfrac{1}{2}\xi)B_2^{11}(s,t). \tag{12}$$

This yields $A=-(F_A\gamma/bf_\pi m_A^2)(1+\frac{1}{2}\xi)(1+C)B_1^{11}(s,t)$ and a form for $F_\pi(t)$ depending on our choice for A_n. A particularly simple choice occurs in the approximation of neglecting terms of order $m_\pi^2/2m_\rho^2$: $a_3=\xi$, and $C=a_n=0$, $n>3$. Then $F_\pi(t)\sim\Gamma(1-\alpha_t)/\Gamma(\frac{5}{2}-\alpha_t)$ as proposed by Oyanagi,[3] but following now for arbitrary ξ.

Our discussion has been restricted to the single-π- and -A_1-pole model. If Veneziano forms are to be taken seriously we might expect to find sequences of π and A mesons whose poles should be included if we wish to treat the intermediate range for t.[6] Such a discussion is beyond the scope of this paper but it appears that the arbitrariness of $F_\pi(t)$ will still be maintained as long as we require that the matrix element in Eq. (4) has no subtractions in p^2 or k^2. In any case, unless we impose some symmetry constraint on $F_\pi(t)$ such as proposed by Suura,[6] $F_\pi(t)$ will now depend on many unknown inelastic amplitudes involving the heavy π and A mesons.

This paper would have been impossible to write without the valuable discussions the author has had with J. Rosner and H. Suura at Minnesota and with the summer group at the Stanford Linear Accelerator. The author would also like to thank the latter for their hospitality extended to him in July, 1969, when this study was begun.

*Work supported in part by the U. S. Atomic Energy Commission under Contract No. AT-(1-11)-1764.

[1]J. Schnitzer, Phys. Rev. Letters 22, 1154 (1969); R. Arnowitt, P. Nath, Y. Srivastava, and M. H. Friedman, Phys. Rev. Letters 22, 1158 (1969).

[2]G. Veneziano, Nuovo Cimento 57A, 190 (1968); J. Shapiro and J. Yellin, University of California Lawrence Radiation Laboratory Report No. UCRL-18500 (unpublished); J. A. Shapiro, Phys. Rev. 179, 1345 (1969); C. Lovelace, Phys. Letters 28B, 264 (1968); C. J. Goebel, M. L. Blackmon, and K. C. Wali, Phys. Rev. 182, 1487 (1969).

[3]Y. Oyanagi, University of Tokyo Reports Nos. UT-16, 1969, and UT-19, 1969 (to be published).

[4]J. L. Rosner and H. Suura, Phys. Rev. (to be published).

[5]In an August 1969 preprint, P. Nath, R. Arnowitt, and M. H. Friedman also discuss this problem but unnecessarily restrict the s,t form of their subtraction functions.

[6]H. Suura, Phys. Rev. Letters 23, 551 (1969).

UNSTABLE PARTICLES, TWO-BODY INELASTIC UNITARITY, AND VENEZIANO'S MODEL*

N. F. Bali, Darryl D. Coon, and Jan W. Dash
Department of Physics, University of Washington, Seattle, Washington 98105
(Received 30 June 1969)

We propose an integral over the Veneziano amplitude which introduces singularities of two-body inelastic unitarity in the amplitude. The model provides a framework for a unified treatment of quantization conditions, the stability of particles, the Pomeranchuk singularity, and decay of particles by pion emission.

Recently a model for the strong amplitude has been suggested by Veneziano[1] which has a number of desirable properties but is in disagreement with unitarity. In this note we propose a generalization which partially remedies this difficulty. We take the view that the Veneziano amplitude is in some sense a zeroth approximation to the correct amplitude, and its success suggests that the corrections are small. We write the Veneziano amplitude as[2]

$$A_V(s,t)=\frac{\Gamma(J^{\min}-\alpha(s))\Gamma(J^{\min}-\alpha(t))}{\Gamma(2J^{\min}+l-\alpha(s)-\alpha(t))}, \tag{1}$$

where $J^{\min}$ is the lowest physical angular momentum on α, and l is an appropriate integer ≤ 0 and $J^{\min} \geq |l|$. We take $\alpha(y)=a+by$. The fundamental Ansatz of our model is

$$\begin{aligned} A(s,t)=\int_{-\infty}^{L} da\,\rho(a,a^0,L,\epsilon,\cdots)\Bigg\{&\frac{\Gamma(J^{\min}-a-bs)\Gamma(J^{\min}-a-bt)}{\Gamma(l+2J^{\min}-2a-bs-bt)}\\ &-\sum_{n=0}^{k-1}\frac{(-1)^n}{n!}\left[\frac{\Gamma(J^{\min}-a-bt)}{\Gamma(l+J^{\min}-a-bt-n)(J^{\min}-a-bs+n)}+\frac{\Gamma(J^{\min}-a-bs)}{\Gamma(l+J^{\min}-a-bs-n)(J^{\min}-a-bt+n)}\right]\Bigg\}\\ &+\sum_{n=0}^{k-1}\frac{(-1)^n}{n!}\left[\frac{\Gamma(J^{\min}-a^0-bt)}{\Gamma(l+J^{\min}-a^0-bt-n)(J^{\min}-a^0-bs+n)}+\frac{\Gamma(J^{\min}-a^0-bs)}{\Gamma(l+J^{\min}-a^0-bs-n)(J^{\min}-a^0-bt+n)}\right], \end{aligned} \tag{2}$$

which amounts to "smearing" with a weight ρ all but k poles of the Veneziano amplitude about the intercept of their trajectory. Clearly the above form is not the most general one[3] but has the virtue of being quite simple and yet possessing a number of very desirable features. If the deviations from the Veneziano form of Eq. (1) are small, our generalization should be reasonably good. Note that one can write partial-fraction (pole) expansions for the Veneziano form in the integrand of Eq. (2) and see that Eq. (2) has the form of a dispersion integral in either s or t with ρ being a weight function. Although the form of