ISOTOPIC-SHIFT PARAMETERS IN THE NILSSON MODEL

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(Received 3 June 1969)

Fradkin's "compressibility under deformation" parameter ξ is calculated using the deformed Nilsson model of the nucleus. Numerical values in the range $-0.21 \leq \xi \leq +0.26$ are obtained for ${}_{62}$Sm, ${}_{64}$Gd, ${}_{70}$Yb, and ${}_{92}$U and for $-0.1 \leq \beta_\omega \leq +0.3$, to be compared with the empirical value $\xi \sim -0.2 \pm 0.1$. It is suggested that the effect comes from the saturation property of the nuclear force.

Various methods have been developed over the past few years to study accurately small differences in nuclear charge and mass distributions. Muonic atomic transitions,[1] elastic electron scattering,[2] isomeric shifts,[3] and isotopic shifts of atomic spectral lines[4] have supplied information on the proton-distribution rms radii, while stripping reactions[5] and neutron scattering[6] have studied the overall mass distribution. In particular, Stacey[4] has described the various attempts made to explain the changes in $\langle R_p^2\rangle$, the mean-square charge radius, for neighboring isotopes in terms of the changes in the atomic number A and β_p, the proton-distribution deformation parameter. More specifically it is well known that Fradkin[7] was led to introduce empirically a "compressibility under deformation" parameter ξ in order to account quantitatively for the β_p dependence of the isotopic-shift effect. Following a paper by the present authors[8] (later referred to as CC), where a theoretical justification for introducing such a parameter was given in the framework of the potential-volume-conserving deformed-oscillator model, the present Letter describes a detailed numerical calculation of ξ using the deformed Nilsson model.

The formalism of Bohr[9] has been generally used by many workers to investigate isotopic shifts in heavy deformed nuclei.[7,10] We shall follow CC and use a slight but convenient modification of it. We suppose that the nuclear wave function obeys the well-known assumption of factorization into a rotational part and an intrinsic deformed part $|\chi_0\rangle$. In the principal-axis system we have

$$\langle R_p^2\rangle = \langle\chi_0|X_p^2+Y_p^2+Z_p^2|\chi_0\rangle = \sum_{k=1}^{3}\langle X_{kp}^2\rangle. \quad (1)$$

The deformation parameters β_p and γ_p of the proton distribution are defined through the equations

$$\langle X_p^2\rangle_a = [\langle X_{1p}^2\rangle\langle X_{2p}^2\rangle\langle X_{3p}^2\rangle]^{1/3}, \quad (2)$$

$$\langle X_{kp}^2\rangle = \langle X_p^2\rangle_a \times \exp[2(5/4\pi)^{1/2}\beta_p\cos(\gamma_p - \tfrac{2}{3}k\pi)]. \quad (3)$$

Introducing (3) in (1) and expanding the exponentials for small β_p yields the usual relation

$$\langle R_p^2\rangle \cong 3\langle X_p^2\rangle_a[1+(5/4\pi)\beta_p^2+\mathcal{O}(\beta_p^3,\gamma_p)+\cdots]. \quad (4)$$

The quantity $\langle X_p^2\rangle_a$ was found empirically by Fradkin to have an additional dependence on β_p of the form

$$\langle X_p^2\rangle_a \cong \langle X_p^{2*}\rangle[1+\xi\beta_p^2], \quad (5)$$

where $\xi \sim -5/8\pi = -0.2$, and $\langle X_p^{2*}\rangle$ is independent of β_p, whereas for a strictly incompressible density distribution it is $\langle X_p^2\rangle_a$ which is independent of β_p. Our task is simply to compute $\langle X_p^2\rangle_a$ accurately and then determine its dependence on β_p.

The deformed Nilsson model[11] assumes that the intrinsic wave function $|\chi_0\rangle$, corresponding to a $K=0$ state of an even-even nucleus, is obtained as an eigenstate of the one-body Hamiltonian

$$H_{\rm Nils} = H^0(\beta_\omega,\gamma_\omega) + H^1_{\beta_\omega,\gamma_\omega}(\vec{1}'\cdot\vec{\sigma}, \vec{1}'\cdot\vec{1}'). \quad (6)$$

Here H^0 is the Hamiltonian of a deformed oscillator and is written as

$$H^0(\beta_\omega,\gamma_\omega) = \sum_{k=1}^{3}\left(\frac{p_k^2}{2m} + \tfrac{1}{2}m\omega_k^2 r_k^2\right), \quad (7)$$

with

$$\hbar\omega_k = \hbar\omega_0\exp[-(5/4\pi)^{1/2}\beta_\omega\cos(\gamma_\omega - \tfrac{2}{3}k\pi)], \quad (8)$$

and

$$\hbar\omega_0 \cong E_0/A^{1/3}, \quad E_0 \approx 41 \text{ MeV}.$$

The form (8) ensures that $\omega_1\omega_2\omega_3=\omega_0^3$, a potential-volume-conserving condition proposed by Mottelson[12] and used recently by Lande[13] to enforce the saturation property of the model. The difference $\Delta H^0(\beta_\omega,\gamma_\omega)=H^0(\beta_\omega,\gamma_\omega)-H^0(0,0)$ is to be viewed as the quadrupole interaction plus a small amount of repulsive monopole interaction. The monopole part leads to an effective change in ω_0 of the form

$$\omega_0'=\omega_0+\Delta\omega_0=[\tfrac{1}{3}(\omega_1^2+\omega_2^2+\omega_3^2)-\omega_0^2]^{1/2} \cong \omega_0[1+(5/8\pi)\beta_\omega^2+O(\beta_\omega^3,\gamma_\omega)+\cdots]. \quad (9)$$

We shall come back to this interesting equation later. The parameter β_ω is approximately equal to Nilsson's δ when $\gamma_\omega=0$. There is some freedom in the choice of $H^1_{\beta_\omega,\gamma_\omega}(\vec{l}'\cdot\vec{\sigma},|\vec{l}'|^2)$ when $\beta_\omega\neq 0$. Here we choose to use[14]

$$H^1_{\beta_\omega,\gamma_\omega}(\vec{l}'\cdot\vec{\sigma},|\vec{l}'|^2)=-\hbar\omega_0\chi[\vec{l}'\cdot\vec{\sigma}+D\vec{l}'\cdot\vec{l}'], \quad (10)$$

where

$$l_k'=\frac{1}{\omega_0}\sum_{i,j=1}^{3}\epsilon_{ijk}\omega_i r_i p_j, \quad k=1,2,3. \quad (11)$$

The form (10) reduces to the usual one when $\beta_\omega=0$, and in general (7) can always be diagonalized exactly inside a single deformed major shell with quantum number N. We assume the values of χ and D to be independent of β_ω and γ_ω and we choose them separately for each deformed major shell. For $N=0$ to 6 the following values are used:

$$\chi_N=\{0, 0.1, 0.1, 0.07, 0.05, 0.05, 0.05\},$$

$$D_N=\{0, 0, 0, 0.35, 0.45, 0.45, 0.45\}.$$

Using $b_0^2=\hbar/m\omega_0$, the equivalent spherical-oscillator radius parameter, the dimensionless proton quadrupole moments Q_i, $i=1,2,3$, can be defined in a Cartesian basis as

$$Q_i=\langle\chi_0|\textstyle\sum_p(r_i^2/b_0^2)|\chi_0\rangle, \quad (12)$$

with

$$\langle X_{kp}^2\rangle=b_0^2Q_i. \quad (13)$$

The sum is over the occupied single-particle proton states $|\varphi_p\rangle$, chosen as having the lowest value of $\langle\varphi_p|\frac{3}{4}H^0+\frac{1}{2}H^1|\varphi_p\rangle$ which is taken as an approximation to the Hartree energy. The deformation parameters β_p and γ_p corresponding to the proton density distribution can now be computed by using (2) and (3) and noting that

$$\langle X_p^2\rangle_a=b_0^2Q_{pa}=(\hbar/m\omega_0)Q_{pa}, \quad (14)$$

with $Q_{pa}=[Q_1Q_2Q_3]^{1/3}$. In the present work we have calculated $Q_{pa}\beta_p$ vs β_ω for $-0.1\leq\beta_\omega\leq+0.3$ ($\gamma_\omega=0$ or $60°$) by obtaining the eigenfunctions and eigenvalues of (6) to an accuracy of 1 part in 10^6. The study included all even-Z cases up to 100. The results are presented in Table I for the deformed nuclei $_{62}$Sm, $_{64}$Gd, and $_{70}$Yb, for which data exist,[15] and for $_{92}$U by comparison. The quantity $\Delta Q_{pa}=Q_{pa}(\beta_\omega)-Q_{pa}(0)$ is given together with β_p ($\gamma_p=0$ or $60°$). Fradkin's parameter ξ is then readily computed as

$$\xi(\beta_p)=\Delta Q_{pa}/Q_{pa}\beta_p^2. \quad (15)$$

The values Q_{pa} change abruptly when the set of occupied orbits with lowest energy changes.[11] For the first three cases in Table I the sudden increase in ΔQ_{pa} comes from occupying orbits in the next major shell, which has a larger rms radius. The occurrence of $\Delta Q_{pa}=0$ for $\beta\leq 0.15$ in $_{70}$Yb is due to the fact that the protons in this case close a major oscillator shell ($N=4$). These particular results, in the region of strong deformation, indicate that ξ fluctuates between approximately ±0.25. The results for all the other nuclei (not given here), however, show that ξ tends to be negative and of the order of -0.15. Since the precise value of ξ is clearly model dependent it is probable that a refitting of isotopic data using the computed values of ξ, as was done here, instead of $-5/8\pi$, would prove a sensitive test of the Nilsson model and its parameters. We should also mention that the present results could be useful in other related fields. In order to compare the results of isomeric-shift measurements, which give $\Delta\langle R_c^2\rangle/\langle R_c^2\rangle$, with recent Coulomb excitation measurements[16] of $\Delta B(E2)/B(E2)$, it is important to establish the relationship between $\Delta\langle R_c^2\rangle\sim\langle R_p^2\rangle$ and $\Delta\beta_p^2$. Usually ξ has been taken to be zero for the comparison. This relation is also important for predicting $\Delta\langle R_p^2\rangle/\langle R_p^2\rangle$ from β-band mixing theory. In $^{152}_{62}$Sm, this calculation was done[17] with $\beta=0.304$ and $\xi=0$, in good agreement with the results of Table I.

Finally, we conclude with a short discussion of the origin of this nuclear compressibility factor ξ. We may find a heuristic explanation of the effect by going back to Eq. (9). Had we decided not to include $\Delta N=\pm2,\pm4,\cdots$ effects in the diagonalization of $H_{\rm Nils}$ we would at least have used ω_0' instead of ω_0 in (14). Since ω_0' occurs in the denominator the whole calculation would have started with the expression

$$\langle R_p^2\rangle=\frac{3\hbar^2}{m\omega_0}\left(1-\frac{5}{8\pi}\beta_\omega^2\right)Q_{pa}'\left(1+\frac{5}{4\pi}\beta_p^2\right). \quad (16)$$

Table I. Changes in Q_{pa} vs β_p and β_ω for Sm, Gd, Yb, and U.

	$_{62}$Sm		$_{64}$Gd		$_{70}$Yd		$_{92}$U	
β_ω	β_p	ΔQ_{pa}	β_p	ΔQ_{pa}	β_p	ΔQ_{pa}	β_p	ΔQ_{pa}
0.3	0.280	-0.037	0.273	0.089	0.332	1.206	0.289	-0.520
0.275	0.266	0.060	0.258	0.107	0.234	0.929	0.274	-0.478
0.25	0.211	-0.273	0.244	0.126	0.219	0.950	0.219	-0.524
0.225	0.198	-0.269	0.229	0.146	0.204	0.972	0.204	-0.502
0.2	0.185	-0.265	0.162	-0.143	0.189	0.995	0.189	-0.476
0.175	0.172	-0.260	0.148	-0.140	0.134	0.575	0.174	-0.443
0.15	0.158	-0.254	0.135	-0.136	0.075	0.000	0.158	-0.405
0.125	0.145	-0.246	0.122	-0.132	0.062	0.000	0.140	-0.358
0.1	0.130	-0.236	0.108	-0.126	0.050	0.000	0.122	-0.302
0.075	0.087	-0.092	0.065	-0.027	0.037	0.000	0.102	-0.237
0.05	0.068	-0.066	0.046	-0.015	0.025	0.000	0.080	-0.166
0.025	0.046	-0.040	0.024	-0.004	0.012	0.000	0.057	-0.102
0	0.000	0	0.000	0	0.000	0	0.016	0
-0.025	-0.038	-0.024	-0.026	-0.006	-0.012	0.000	-0.036	-0.017
-0.05	-0.064	-0.050	-0.055	-0.033	-0.025	0.000	-0.068	-0.092
-0.075	-0.093	-0.107	-0.084	-0.081	-0.037	0.000	-0.086	-0.122
-0.1	-0.120	-0.171	-0.111	-0.136	-0.050	0.000	-0.104	-0.159
$Q_{pa}(\beta_\omega=0)$	90.333		94.000		105.000		152.652	

Now this expression already includes the Fradkin correction (assuming $\beta_\omega \sim \beta_p$); it would be therefore sufficient to show that Q_{pa}' has a weak dependence on β_p. This argument suggests that Fradkin's parameter is the result of the saturation properties of the nuclear force. This second approach might also allow one to include more easily the effects of pairing,[18] a point which was neglected here. Further work in this direction will be reported elsewhere.

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ISOSPIN SUM RULES IN NUCLEAR PHYSICS*

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A systematic analysis of the electric dipole sum rules in the different isospin channels is given. The connections between these sum rules and the well-known old ones are classified. A simple application is given in resolving the isospin components of the nuclear giant resonance.

The Cabibbo-Radicati[1] sum rule has been considered one of the most useful results of current-algebra theory. It has been derived for isospins $T=\frac{1}{2}$ and $T=1$; these cases practically exhaust elementary-particle applications.

This sum rule has been applied in nuclear physics[2] to the isodoublets ^{3}H and ^{3}He and with various models to other nuclei.[3] Recently[4] a generalization of this sum rule to any T has been given which is in contrast to a previous generalization given by the authors.[3]

The aim of this Letter is to illustrate clearly the connections between the generalized Cabibbo-Radicati sum rule,[3] the sum rule recently proposed,[4] and the other sum rules, some of which are well known,[5] which one can deduce in a simple way from the algebra of the dipole operator D. We shall give all these sum rules in a compact formulation, in the long-wavelength approximation. In the same manner we exploit other possibilities which one can systematically obtain if, besides the algebra on the D operators, the commutators $[D,H]$ are also known. As an illustration of the utility of these sum rules we discuss some applications to the analysis of the isospin components of the giant resonances in nuclei with $T=\frac{1}{2}$.

In order to obtain these sum rules we first perform an isospin analysis of the photoreactions. For simplicity we treat the dipole excitation of a target $|TT_z\rangle$, but the same techniques may be applied to any isovector operator. We start with the formal definition of the transition strength,

$$f_{0n}{}^{ab}=4\alpha\pi^2\langle 0|D^a|n\rangle\langle n|D^b|0\rangle\,\omega_n,$$

where $a,b=3,+,-$; $|0\rangle$ and $|n\rangle$ are the initial and final states of the target; $\omega_n=E_n-E_0$; and $D^a=\sum\frac{1}{2}\tau_{ai}\times x_i$. (For $a=b=3$ we obtain the electric dipole transition strength; for $a=+$, $b=-$ we obtain the first forbidden Fermi transition strength.) So we have symbolically

$$\sum f_{0n}{}^{ab}\omega_n{}^q=\int\sigma_{ab}(\omega)\omega^q d\omega,$$

with obvious notation. In particular we have for $q=-1$

$$\int(\sigma_{ab}/\omega)d\omega=4\pi^2\alpha\langle 0|D^aD^b|0\rangle.$$

In order to derive the sum rule it is useful to separate the isoscalar, isovector, and isotensor contributions of $\int(\sigma_{ab}/\omega)d\omega$ and to express these three parts as functions of the reduced cross sections in