Center at Cornell University, Report No. 1206.
[1]L. H. Nosanow and W. J. Mullin, Phys. Rev. Letters 14, 133 (1965); R. A. Guyer and L. I. Zane, Phys. Rev. (to be published).
[2]I. Pomeranchuk, Zh. Eksperim. i Teor. Fiz. 20, 919 (1950).
[3]L. Goldstein, Phys. Rev. 159, 120 (1967).
[4]Yu. D. Anufriyev, Zh. Eksperim. i Teor. Fiz.—Pis'ma Redakt 1, No. 6, 1 (1965) [translation: JETP Letters 1, 155 (1965)].
[5]R. T. Johnson, R. Rosenbaum, O. G. Symko, and J. C. Wheatley, Phys. Rev. Letters 22, 449 (1969).
[6]J. R. Sites, J. R. Eckardt, and D. M. Lee, Bull. Am. Phys. Soc. 13, 1669 (1968); J. R. Eckardt, thesis, Cornell University, 1969 (unpublished).
[7]J. L. Baum, D. F. Brewer, J. G. Daunt, and D. O. Edwards, Phys. Rev. Letters 3, 127 (1959).
[8]O. E. Vilches and J. C. Wheatley, Rev. Sci. Instr. 37, 819 (1966).
[9]R. L. Mills, E. R. Grilly, and S. G. Sydoriak, Ann. Phys. (N.Y.) 12, 41 (1961); Ref. 7, as normalized by A. C. Anderson, W. Reece, and J. C. Wheatley, Phys. Rev. 130, 1644 (1963).
[10]A. G. Anderson and A. G. Redfield, Phys. Rev. 116, 583 (1959).
[11]G. S. Rushbrooke and P. J. Wood, Mol. Phys. 1, 257 (1959).
[12]E. M. Saunders, Phys. Rev. 126, 1724 (1962); R. L. Garwin and A. Landesman, Physics 2, 107 (1965); D. J. Thouless, Proc. Phys. Soc. (London) 86, 893 (1965).
[13]M. E. Fisher and M. F. Sykes, Physica 28, 939 (1962).
[14]In Ref. 5, the authors indicate that the data of A. C. Anderson, W. Reese, and J. C. Wheatley, Phys. Rev. Letters 7, 366 (1961) can be analyzed to give $T_N = 2.4 \pm 0.3$.
[15]M. F. Panczyk, R. A. Scribner, G. C. Straty, and E. D. Adams, Phys. Rev. Letters 19, 1102 (1967).
[16]R. C. Richardson, E. Hunt, and H. Meyer, Phys. Rev. 138, A1326 (1965); as analyzed by H. Meyer, J. Appl. Phys. 39, 390 (1968).
[17]M. F. Pancyk and E. D. Adams, Phys. Rev. (to be published).

TRAPPED-PARTICLE INSTABILITY*

W. L. Kruer and J. M. Dawson
Plasma Physics Laboratory, Princeton University, Princeton, New Jersey 08540

and

R. N. Sudan†
School of Electrical Engineering and Center for Radiophysics and Space Research, Cornell University, Ithaca, New York 14850
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A simple model for a new instability resulting from particles trapped in a large-amplitude electrostatic wave is invoked to explain the generation of satellite frequencies in a recently reported experiment by Wharton, Malmberg, and O'Neil. The model predicts satellites on the large-amplitude wave at a frequency separation proportional to $\sqrt{E}$, where E is the amplitude of the large wave. The predicted growth rates reasonably account for the observed growth of these satellites.

In a recent paper, Wharton, Malmberg, and O'Neil[1] describe the excitation of large-amplitude plasma waves by means of a probe immersed in a plasma having an electron temperature of 9.4 eV, an electron density of 5×10^8 cm^{-3}, and relatively cold ions. Apart from observing the reduction in the damping rate for large-amplitude waves predicted by O'Neil[2] and Al'tshul and Karpman,[3] the experiment also showed the growth of sidebands to the frequency of the large-amplitude wave. The frequency separation of these satellite bands was found to be proportional to the square root of the wave amplitude. This indicates that trapped particles bouncing in the potential trough of the wave at a frequency $\omega_B = (eEk/m)^{1/2}$ (E is the wave field, k the wave number) must play an important role in the generation of these sidebands. In addition, a broadening of the frequency of the large-amplitude wave was observed.

We propose the following explanation for these observations. A significant number of particles are trapped in the trough of the wave due to its large amplitude. Because the electrostatic fields are largest in the vicinity of the probe, most of the trapping takes place there. These particles then move with the wave, with a mean velocity equal to its phase velocity. The trapped particles oscillate in the wave trough. The frequency of this oscillation is roughly constant for a large number of the trapped particles, varying only a few percent for all those particles trapped within $\frac{1}{6}$ of a wavelength (±30°) from the bottom of the trough. Thus, these particles should be able to act coherently (like a beam) and we might expect something similar to a two-stream instability. We propose that this instability is responsible for the sidebands observed by Wharton, Malmberg, and O'Neil.

To investigate the possibility, we approximate the trapped particles by using a bunched beam of harmonic oscillators. For simplicity, we assume that the particles have a commom oscillation frequency and that they are grouped at the bottoms of the wave troughs. The equation of motion for such an oscillator, perturbed by an electric field (not the electric field of the large-amplitude wave), is

$$\ddot{x}_n(t) = -\omega_B{}^2(x_n - x_{n0} - v_p t) - \frac{e}{m}\int \frac{E(k^1, \omega^1)e^{ik^1x_n - i\omega^1 t}}{(2\pi)^2}dk^1 d\omega^1. \tag{1}$$

Here v_p is the phase velocity of the large-amplitude wave, $x_n - x_{n0} - v_p t$ is the position of the oscillator relative to the nth trough, ω_B is the oscillator frequency and the bounce frequency in the large wave, and $E(k, \omega)$ is the Fourier amplitude of the perturbing field. (ω_B mocks up the effects of the E field associated with the large-amplitude wave.) Fourier analysis in time of the displacement of an oscillator $\xi_n(t) = x_n(t) - x_{n0} - v_p t$ gives

$$\xi_n(\omega) = \frac{e}{m(\omega^2 - \omega_B{}^2)}\int \frac{E(k^1, \omega + k^1 v_p)e^{ik^1 x_{n0}}}{2\pi}dk^1. \tag{2}$$

To derive a dispersion relation we demand that the electric field produced by the perturbed motion of the oscillators be consistent with the field perturbing the oscillators. The charge density due to the (assumed small) perturbation of an oscillator is

$$\rho_n(x, t) = e\xi_n(t)\frac{\partial}{\partial x}\delta(x - x_{n0} - v_p t). \tag{3}$$

Fourier analyzing and summing over all such oscillators, we obtain

$$\rho(k, \omega) = ike\sum_n N_n e^{-ikx_{n0}}\xi_n(\omega - kv_p), \tag{4}$$

where ξ_n is the Fourier-analyzed perturbed position evaluated at $\omega - kv_p$, and N_n is the number of trapped particles in the nth wave trough. Treating the background plasma as a continuous medium with dielectric function $\epsilon_L(k, \omega)$, we have from Poisson's equation

$$ik\epsilon_L(k, \omega)E(k, \omega) = 4\pi\rho(k, \omega). \tag{5}$$

Introducing (2) and (4) into (5) and using the identity[4]

$$\frac{\lambda_0}{2\pi}\sum_n e^{i(k^1 - k)n\lambda_0} = \sum_m \delta(k^1 - k - mk_0),$$

we readily obtain

$$E(k, \omega) = \frac{\omega_T{}^2}{\Omega^2 - \omega_B{}^2}\sum_m \frac{E(k + mk_0, \omega + m\omega_0)}{\epsilon_L(k, \omega)}. \tag{6}$$

In the above, λ_0 is the wavelength of the large amplitude wave, $k_0 = 2\pi/\lambda_0$, ω_T is the plasma frequency of the trapped particles (treating them as being spread out over a wavelength), and $\omega_0 = k_0 v_p$.

Equation (6) represents a coupled set of equations for the Fourier amplitude of one mode in terms of the Fourier amplitudes of other modes. We simplify this set by observing the following. The plasma does not support wave-type solutions for ω greatly different from the plasma frequency, ω_p. For $\omega \approx \omega_p$ (where we expect the plasma to oscillate), the two dominant waves are $E(k, \omega)$ and $E(k - 2k_0, \omega - 2\omega_0)$. The frequency of the large-amplitude wave is ω_0 (also nearly ω_p). Keeping only these two terms, we obtain two coupled equations for $E(k, \omega)$ and $E(k - 2k_0, \omega - 2\omega_0)$. The condition for a solution to exist is that the determinant of their coefficients vanishes. The resulting dispersion relation is

$$1 = \frac{\omega_T{}^2}{\Omega^2 - \omega_B{}^2}\left[\frac{1}{\epsilon_L(k, \omega)} + \frac{1}{\epsilon_L(k - 2k_0, \omega - 2\omega_0)}\right]. \tag{7}$$

To investigate this dispersion relation we adopt the warm-fluid approximation for $\epsilon_L(k, \omega)$, the dielectric function of the background plasma:

$$\epsilon_L(k, \omega) = 1 - \frac{\omega_p{}^2}{\omega^2 - 3k^2 v_T{}^2},$$

where v_T is the thermal velocity of the background plasma. The dispersion relation then reduces to a polynomial. The roots of this polynomial are obtained numerically by the Newton-Raphson iterative technique. Results for parameters typical of experimental values are displayed in Figs. 1 and 2.

There appears an instability near ω_0 and k_0. The growth rate as a function of k is shown in

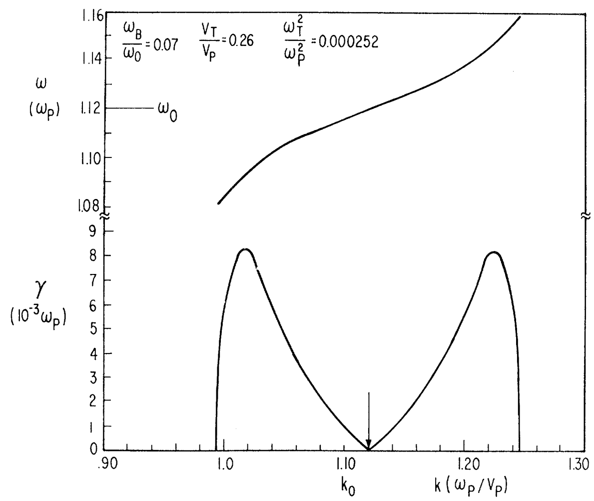

FIG. 1. Growth rate and frequency as functions of wave number.

Fig. 1. It forms a double-humped curve about k_0: On each side of k_0 it rises rather sharply from zero to a maximum and then falls off abruptly. The frequency as a function of k is given by the upper curve in this figure. We interpret the rather sharp maxima in the growth rate as producing the upper and lower sidebands observed in the experiment.

The upper curve in Fig. 2 shows the growth rate as a function of the bounce frequency. These growth rates reasonably account for the observed production of the sidebands. For example, in the experiment for $\omega_B \simeq 0.07\omega_0$ and $v_T \simeq 0.26v_p$, the sideband grew by a factor of 10 in a time of roughly $225\omega_p^{-1}$, giving $\gamma \simeq 5\times10^{-3}\omega_p$. This compares favorably with this model's prediction of $\gamma \simeq 8\times10^{-3}\omega_p$. Furthermore, the slight growth rates of adjacent frequencies in our model can account for the observed "fuzzing" of the frequency of the large-amplitude wave.

The lower curve in Fig. 2 gives the frequency separation of the sidebands from the large-amplitude wave. This frequency separation is reasonably linear with the square root of the large-wave amplitude in agreement with experiment. However, the magnitude of this frequency separation agrees only roughly with the observations. The experiment yields a separation about equal to the bounce frequency of the electrons in the large-amplitude wave. The experimental uncertainty in this measurement is about 40%, since the electric field amplitude is measured to within a factor of 2. For the experimental parameters, our simple model yields a frequency separation of about 35% of the bounce frequency. We do not believe that this is a serious discrepancy, since it can be explained by incorporating the finite size of the plasma and making allowance for the uncertainty in the number of trapped particles.

For comparison with the experiment we can only estimate the number of trapped particles, which we have done by using the undisturbed Maxwellian distribution and the experimentally measured phase velocity and wave amplitude. Reasonable uncertainties in these parameters lead to a roughly 10-20% uncertainty in the frequency separation predicted by our simple model.

More important is the finite size of the plasma in the real experiment. Our analysis has assumed an infinite uniform plasma. In the experiment the wavelength of the large-amplitude wave is about 2.5 cm while the effective radius of the plasma is about 1 cm. To incorporate the role

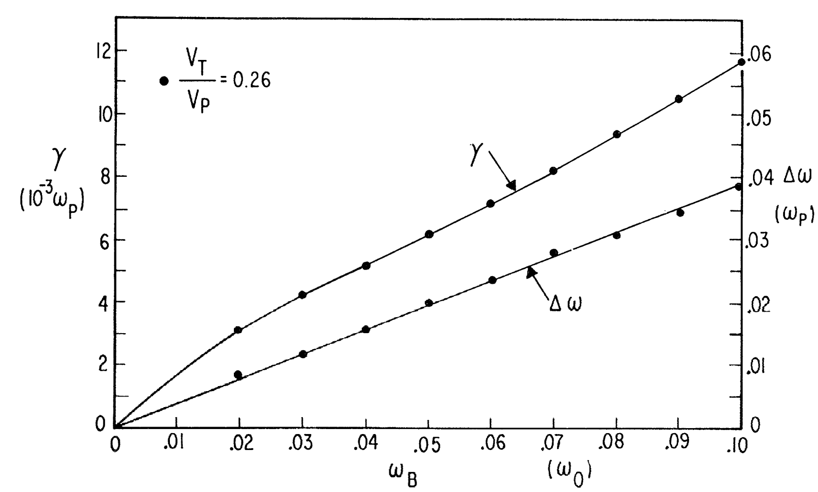

FIG. 2. Growth rate and frequency separation as functions of the bounce frequency.

of the finite size we mocked up the experimentally determined dispersion relation[5] with a very simple relation,

$$\epsilon_{FL}(k,\omega)=1-\frac{\omega_{p\,\mathrm{eff}}^{\;2}}{\omega^2-3k^2v_T^{\;2}},$$

where the effective plasma frequency is defined by

$$\omega_{p\,\mathrm{eff}}^{\;2}=\omega_p^{\;2}\frac{k^2R^2}{1+k^2R^2}.$$

R is chosen to give a good fit of this function to the experimentally determined dispersion relation in the neighborhood of ω_0 and k_0.

Figure 3 exhibits the results for the parameters of Fig. 1. The sideband structure is still evident, but now the two peaks are more distinctly separated. It is readily seen that the frequency separation is significantly increased; it is now 60% of the bounce frequency, if one chooses the peak growth rate, and even larger if one takes a farther separated wave as characteristic of the broader band of unstable waves. The growth rates, on the other hand, are roughly 20% greater than the infinite-size example.

We note that our simple model predicts a sideband on both sides of the central frequency in agreement with the experiment. However, we understand[6] that the upper sideband was somewhat suppressed relative to the lower. Landau damping may account for this suppression. The upper sideband has a phase velocity less than that of the large wave (larger ω but even larger k), while the lower sideband has a larger phase velocity (lower ω but even lower k). Of course, the Landau damping may be strongly influenced by the large-amplitude wave, so this aspect should be more throughly investigated.

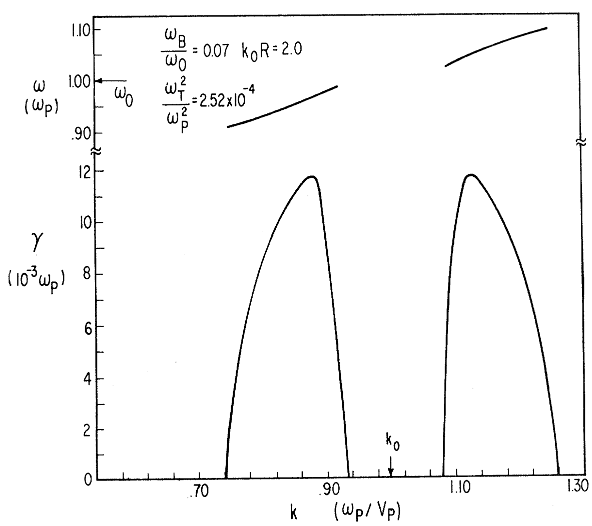

FIG. 3. Growth rate and frequency as functions of wave number (finite size system).

Finally, we have investigated this instability in some detail with computer experiments on the one-dimensional sheet model.[7] These results, in reasonable agreement with both the real experiment and the simple calculation, will be presented in a forthcoming paper. Goldman has recently obtained this instability via a more general formalism.[8] His treatment is very valuable for exploring this instability under more general conditions.

In conclusion, we have presented a simple model which by no means exhausts this problem. However, we have been able to incorporate in a simple physical model the role of particle trapping, and to predict the essential features of experiment. We believe that the physical insight derived from this simple model can facilitate the inclusion of trapping effects in more detailed treatments.

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[1]C. B. Wharton, J. H. Malmberg, and T. M. O'Neil, Phys. Fluids 11, 1761 (1968).

[2]T. M. O'Neil, Phys. Fluids 8, 2255 (1965).

[3]L. M. Al'tshul and V. I. Karpman, Zh. Eksperim. i Teor. Fiz 49, 515 (1965) [translation: Soviet Phys.—JETP 22, 361 (1966)].

[4]M. J. Lighthill, Fourier Analysis and Generalised Functions (Cambridge University Press, London, England, 1964).

[5]J. H. Malmberg and C. B. Wharton Phys. Rev. Letters 19, 775 (1967).

[6]C. B. Wharton, private communications.

[7]W. L. Kruer, thesis, Princeton University, 1969 (unpublished).

[8]M. Goldman, University of California, Los Angelos Physics Department Report No. R-52, 1969 (unpublished).