SIMPLE PROOF AND GENERALIZATION OF GRIFFITHS' SECOND INEQUALITY

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A recent generalization by Sherman of Griffiths' second inequality on correlations in Ising ferromagnets is further generalized. Straightforward proofs are provided both for the original inequality and its generalization.

Recently, Griffiths[1] obtained remarkable inequalities for the correlation functions of Ising ferromagnets with two-body interactions. These inequalities were subsequently generalized by Kelly and Sherman to systems with interactions involving an arbitrary number of spins.[2] The second inequality, described below as theorem 1, was then generalized by Sherman.[3] These inequalities have received several applications of physical interest. They have been used to prove the existence of the infinite-volume limit for the correlation functions of Ising ferromagnets,[1] to obtain upper and lower bounds on critical temperatures,[4] to settle the question of the existence of a phase transition in one-dimensional systems with moderately long-range interaction,[5] and to establish rigorous inequalities on critical-point exponents.[6] Their possible extension to systems other than Ising models is under active consideration.[7]

It is therefore of interest to have derivations of the basic inequalitites that are as transparent as possible. In the present paper, we first give a completely straightforward proof of the second inequality, and then a further generalization of Sherman's later result.

We consider a finite set Λ of N sites. Each site carries a spin $\frac{1}{2}$, that is a finite set with two elements called up and down. A configuration of the system is defined by the set of down spins, which is a subset of Λ. Configurations are denoted by capital letters A, B, R, S, etc. The set of configurations Γ is a finite set with 2^N elements. The product RS of two configurations is defined as their symmetric difference $R\Delta S = R\cup S - R\cap S$. With this product, Γ is a commutative finite group. The unit element is the empty set ϕ and every element is of order 2: $R^2=\phi$. With spin r is associated a function σ_r which is 1 for up and -1 for down. The spin products

$$\sigma_R = \prod_{r\in R} \sigma_r$$

are functions on Γ. In fact, they are the characters of the group Γ. They satisfy

$$\sigma_R\sigma_S = \sigma_{RS}, \tag{1}$$

$$\sigma_R(A)\sigma_R(B) = \sigma_R(AB), \tag{2}$$

$$\sigma_R(A) = \sigma_A(R) = (-1)^{n(A\cap R)}, \tag{3}$$

where $n(R)$ denotes the number of sites in R. A physical system is defined by a potential J, which is a real function on Γ, and with which are associated, respectively, a Hamiltonian, a probability density, a partition function, and correlation functions by the formulas

$$H = -\sum_{P\in\Gamma} J(P)\sigma_P, \tag{4}$$

$$W = Z^{-1}\exp(-H), \tag{5}$$

$$Z = \sum_{P\in\Gamma} \exp[-H(P)], \tag{6}$$

$$\langle\sigma_R\rangle = \sum_{A\in\Gamma} \sigma_R(A)W(A). \tag{7}$$

We consider exclusively ferromagnetic systems, by which we mean that $J(R)\geq 0$ for all $R\in\Gamma$. Griffiths' first inequality states that in this case, for all $R\in\Gamma$,

$$\langle\sigma_R\rangle \geq 0. \tag{8}$$

We now turn to the second inequality.

Theorem 1.[1,2] – For all R and S in Γ, the following inequality holds:

$$\langle\sigma_{RS}\rangle - \langle\sigma_R\rangle\langle\sigma_S\rangle \geq 0. \tag{9}$$

Proof. – From (7), we obtain

$$Z^2(\langle\sigma_{RS}\rangle - \langle\sigma_R\rangle\langle\sigma_S\rangle) = \sum_{A,B}[\sigma_{RS}(A) - \sigma_R(A)\sigma_S(B)]\exp[-H(A)-H(B)]. \tag{10}$$

Let $C = AB$. Using (1), (2), and (4), we obtain

$$\cdots = \textstyle\sum_C [1-\sigma_S(C)]\left\{\sum_A \sigma_{RS}(A)\exp\left[\sum_P J(P)[1+\sigma_P(C)]\sigma_P(A)\right]\right\}. \tag{11}$$

The first factor is positive. For fixed C, the quantity in the last brackets is, up to a positive normalization, the average of the spin product σ_{RS} in a new ferromagnetic system associated with potential:

$$J_C(P) = J(P)[1+\sigma_P(C)] \geq 0. \tag{12}$$

The average is positive by Griffiths' first inequality (8), and the theorem is proved.

We now present a more general result.

Theorem 2.—The functions $\omega_{R,S}$ and ρ_P defined by

$$\rho_P(B) = \langle\sigma_B\rangle\langle\sigma_{BP}\rangle, \tag{13}$$

$$\omega_{R,S}(B) = \langle\sigma_B\rangle\langle\sigma_{BRS}\rangle - \langle\sigma_{BR}\rangle\langle\sigma_{BS}\rangle, \tag{14}$$

are positive-definite functions of B for arbitrary fixed P, R, and S in Γ.

We recall that, by definition, a function f on Γ is positive definite if for any complex valued function z on Γ, the following inequality holds:

$$\sum_{A,B\in\Gamma} \bar{z}(A)f(AB)z(B) \geq 0 \tag{15}$$

and that, by Bochner's theorem, a function f is positive definite if and only if its Fourier transform is positive.[8]

Proof of theorem 2.—It is sufficient to prove that the Fourier transforms $\hat{\rho}_P$ and $\hat{\omega}_{R,S}$ defined below are positive:

$$\hat{\rho}_P(T) = \sum_{B\in\Gamma} \sigma_T(B)\langle\sigma_B\rangle\langle\sigma_{BP}\rangle, \tag{16}$$

$$\hat{\omega}_{R,S}(T) = \sum_{B\in\Gamma} \sigma_T(B)(\langle\sigma_B\rangle\langle\sigma_{BRS}\rangle - \langle\sigma_{BR}\rangle\langle\sigma_{BS}\rangle). \tag{17}$$

From (16) and (17), we obtain

$$\hat{\omega}_{R,S}(T) = [1-\sigma_T(R)]\hat{\rho}_{RS}(T) = [1-\sigma_T(S)]\hat{\rho}_{RS}(T). \tag{18}$$

It is therefore sufficient to prove that $\hat{\rho}_P$ is positive. From (7), (16) we obtain

$$\hat{\rho}_P(T) = \sum_{A,B,C} \sigma_B(T)\sigma_B(C)\sigma_B(A)\sigma_P(A)W(A)W(C). \tag{19}$$

We sum over B, using the orthogonality of the characters:

$$\hat{\rho}_P(T) = 2^N\textstyle\sum_A \sigma_P(A)W(A)W(AT) = 2^N Z^{-2}\sum_A \sigma_P(A)\exp\{\sum_Q J(Q)[1+\sigma_Q(T)]\sigma_Q(A)\}. \tag{20}$$

Up to positive normalization factors, the last sum is the average of σ_P in the new ferromagnetic system associated with the potential J_T defined by (12), and is positive by (8). This proves theorem 2.

We now collect a few properties of $\hat{\omega}_{R,S}$, which follow easily from (8), (18), and (20).

Corollary 1.—The function $\hat{\omega}_{R,S}$ defined by (17) has the following properties: (a) $\hat{\omega}_{R,S}(T)=0$ unless $\sigma_T(R)=\sigma_T(S)=-1$. (b) $Z^2\hat{\omega}_{R,S}(T)$ does not depend on the values of $J(Q)$ for those Q for which $\sigma_Q(T)=-1$. (c) For all R, S, T, and Q in Γ and all $n \geq 0$, the following inequality holds:

$$\partial^n[Z^2\hat{\omega}_{R,S}(T)]/\partial J(Q)^n \geq 0. \tag{21}$$

Applying (15) to the function $\omega_{R,S}$, taking z to be the characteristic function of a subgroup Γ_0 of Γ, we obtain the result of Ref. 3.

Corollary 2.—For any subgroup Γ_0 of Γ and any R, S in Γ

$$\sum_{B\in\Gamma_0} (\langle\sigma_B\rangle\langle\sigma_{BRS}\rangle - \langle\sigma_{BR}\rangle\langle\sigma_{BS}\rangle) \geq 0. \tag{22}$$

In particular, for $\Gamma_0 = \{\phi\}$, we recover theorem 1.

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LIQUID STRUCTURE FACTOR OF He^4 BY X-RAY SCATTERING AT SMALL MOMENTUM TRANSFER*

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The liquid structure factor of helium-4 has been determined in the momentum-transfer range 0.2 to 0.8 $Å^{-1}$. At 0.32 K the structure factor exhibits a gentle but decided change of slope near 0.4 $Å^{-1}$ in qualitative agreement with certain theoretical predictions. Below 0.4 $Å^{-1}$ the liquid structure factor agrees with that deduced from the Bijl-Feynman dispersion relation. At 4.99 K and 680 mm pressure we have shown that helium gas is distinctly nonideal.

We have measured the intensity of copper $K\alpha$ x rays scattered from helium-4 as a function of scattering angle both in the gas phase near 5 K and in the liquid phase at several lower temperatures. By normalization against neon at 77 K, structure was observed in the helium gas and a determination of the structure factor of the liquid was made. At. 0.32 K the liquid structure factor shows a gentle but decided change of slope near the momentum transfer 0.4 $Å^{-1}$. This gentle shoulder is in qualitative agreement with the shoulder suggested by Miller, Pines, and Nozières,[1] but is much weaker than the shoulder given by Massey.[2] The liquid structure factor in the range 0.2 to 0.8 $Å^{-1}$ is in good agreement with a recent calculation by Campbell and Feenberg.[3] Our measurements disagree somewhat with those of Gordon, Shaw, and Daunt[4] and also with those of Achter and Meyer.[5]

The Bijl-Feynman dispersion relation[6]

$$E(k)=\hbar^2k^2/2MS(k), \tag{1}$$

where $S(k)$ is the liquid structure factor, k the momentum transfer, and M the mass of helium atom, relates the energy of an elementary excitation in a condensed Bose system to the liquid structure factor at absolute zero. If the excitations are phonons, we expect

$$S(k)=\hbar k/2Mc_1, \tag{2}$$

where c_1 is the velocity of first sound in liquid helium.

It has been shown[7,8] that for finite temperatures the structure factor in the limit of vanishing momentum transfer can be written as

$$\lim_{k\to 0} S(k)\equiv S(0)=nk_BTX_T, \tag{3}$$

where n is the number density, k_B the Boltzmann constant, and X_T the isothermal compressibility. Experiments to date in the momentum transfer range below 0.8 $Å^{-1}$ indicate in some cases[4,9] that $S(k)$ approaches the expected value of nk_BTX_T as $k\to 0$. In other experiments[10] abpve 0.8 $Å^{-1}$ the extrapolated values of $S(0)$ are in close agreement with (3) although the extrapolations themselves are open to errors. In all of these investigations the structure factor was seen to be insensitive to changes in temperature for values of the momentum transfer greater than about 0.8 $Å^{-1}$. This led Jackson[11] and independently Miller, Pines, and Nozières[1] to notice that the experiments did not extrapolate correctly to (2). Miller, Pines, and Nozières suggested that if the data down to 0.8 $Å^{-1}$ were valid at temperatures near absolute zero, then a shoulder should appear on the structure factor curve in the momentum transfer range where the expected linear dependence "joined up" with the measured structure factor. We have investigated the range 0.2 $Å^{-1}<k<0.8$ $Å^{-1}$ in some detail at temperatures as low as 0.32 K to study this problem.