[8]A. Simon, Phys. Rev. 98, 317 (1955).
[9]M. Okabayashi and S. Yoshikawa, Bull. Am. Phys. Soc. 13, 1502 (1968).
[10]D. J. Grove et al., Plasma Physics and Controlled Nuclear Fusion Research (International Atomic Energy Agency, Vienna, Austria, 1969), Vol. I, p. 479.

SUPPRESSION AND ENHANCEMENT OF AN ION-SOUND INSTABILITY BY NONLINEAR RESONANCE EFFECTS IN A PLASMA

B. E. Keen and W. H. W. Fletcher
United Kingdom Atomic Energy Authority, Research Group, Culham Laboratory, Abingdon, England
(Received 18 August 1969)

Results are presented which show the classical anharmonic resonance effects (including hysteresis) on a marginal ion sound instability, when forcing at the fundamental and the subharmonic frequencies. Also, when the instability is well defined it appears to behave as a classical Van der Pol oscillator for drive frequencies near the fundamental and the subharmonics.

There has been considerable interest in the last few years[1-4] in the nonlinear mechanisms which determine the saturation level of a self-excited oscillation (or instability) present in a plasma. Anharmonic effects,[5] mode-mode coupling,[2,3] and wave-particle scattering[6] have been proposed as possible mechanisms in various cases. In particular, the mode-mode coupling approach appears to give rise to the plasma instability behaving as a classical Van der Pol[7] oscillator. We have obtained further experimental evidence that for an ion-sound instability in two different regions the instability behaves (a) as a classical anharmonic oscillator and (b) as a classical Van der Pol oscillator.

Theory.—The instability considered in this case was an $m=0$ ion-sound instability, and the problem has been investigated using the two-fluid approach in slab geometry. The axial magnetic field B_0 is taken along the z direction, and only spatial variations of the form $e^{ik_z z}$ are considered, where k_z is the axial wavelength of the sound instability. The density n is considered of the form $n=n_0+n_1$, where n_0 is the zero-order density and n_1 the perturbed value, and φ_1 and v_1 are taken as the potential and ion-velocity perturbations, respectively. The z component of the electron equation of motion reduces to the form $n_1/n_0=\varphi_1(kT_e/e)$. The ion equation of motion gives

$$\frac{d\vec{v}_1}{dt}=-\frac{e}{M_i}\nabla\varphi_1-\vec{v}_1\nu+\frac{e}{M_i}[\vec{v}_1\times\vec{B}_0], \tag{1}$$

where ν is the ion-neutral collision time and M_i the ion mass. The equation of continuity is given by

$$(dn/dt)+\nabla\cdot(n\vec{v}_1)=S_i, \tag{2}$$

where S_i is a source term due to ionization, etc. caused by large-amplitude oscillations present in the plasma. This source term is taken to be of the form

$$S_i=\alpha n_1-\beta n_1^2-\gamma n_1^3, \tag{3}$$

where $\gamma n_1^2 \ll \beta n_1 \ll \alpha \ll \omega_0=k_z c_s$, and $c_s=(kT_e/M_i)^{1/2}$ is the ion-sound velocity. After eliminating v_1 between (1) and (2), substituting for S_i from (3), and including an external drive term of the form $A\sin\omega t$, the equation reduces to

$$\frac{d^2n_1}{dt^2}+\frac{dn_1}{dt}[\nu-\alpha+2\beta n_1+3\gamma n_1^2]+\omega_0^2 n_1 =\omega_0^2 A\sin\omega t-\nu\beta n_1^2-\nu\gamma n_1^3. \tag{4}$$

This equation may be considered in two situations:

Case (a).—When $\nu>\alpha$, that is, when the self-excited instability is damped out, then the equation is of a standard anharmonic forced-resonance type,[8] which can be written as

$$\frac{d^2n_1}{dt^2}+\omega_0^2n_1=f\left(n_1,\frac{dn_1}{dt}\right)+\omega_0^2A\sin\omega t, \tag{5}$$

where

$$f\left(n_1,\frac{dn_1}{dt}\right) =-\left(\frac{dn_1}{dt}\right)[(\nu-\alpha)+2\beta n_1+3\gamma n_1^2]-\nu\beta n_1^2-\gamma\nu n_1^3.$$

A trial solution is assumed of the form $n_1=b\times\cos(\omega t+\psi)$, where ψ is a phase angle and ω is close to ω_0, and upon substitution Eq. (5) reduces to a cubic equation in b^2 correct to second order. Let $\omega=(\omega_0+\epsilon)$, where $\epsilon\ll\omega_0$, and consider the case where $\gamma b^2 \ll \nu$. Then the equation becomes

$$b^2\left\{\left[2\epsilon-\tfrac{3}{4}\nu\gamma\frac{b^2}{\omega_0}\right]^2+\nu^2\right\}=\omega_0^2A^2. \tag{6}$$

The real roots of the cubic in b^2 give the amplitude of the forced oscillations as ϵ is varied through zero. When A is fairly small, b is fairly small and we have a symmetrical resonance curve as shown in Fig. 1(a) with a maximum at $\epsilon=0$. As A increases the curve changes its shape, but retains its single maximum which moves to positive ϵ, as in Fig. 1(b). When A reaches a certain value A_m, the nature of the curve changes; then Eq. (6) has three real roots corresponding to the region $BCDE$ in Fig. 1(c). The limit of this range is determined by $db/d\epsilon=\infty$ which holds at the points C and D. It may be shown that the dashed part CD corresponds to unstable oscillations, and any small perturbation of the system causes the system to jump from the state C to E or D to B. Hence, if the frequency is gradually increased from A the path $ABC\to EF$ is followed, and if the frequency is decreased starting from F the path $FED\to BA$ is followed, thus showing the hysteresis effect. The maximum amplitude occurs at $db/d\epsilon=0$ and is given by $b_{\max}=\omega_0A/\nu$.

Similarly, if the subharmonic drive proportional to $B\sin\frac{1}{2}(\omega_0+\epsilon)t$ is employed, an equation similar to Eq. (6), where A is replaced by $8\beta B^2/9\omega_0^2$, is obtained, and the maximum is now given by $b_{\max}=8\beta B^2/9\nu\omega_0$ which is proportional to B^2.

Case (b).—Consider the case in Eq. (4) when $\nu\to 0$, then the instability is self-oscillatory and the equation is given by

$$\frac{d^2n_1}{dt^2}-\frac{dn_1}{dt}[\alpha-2\beta n_1-3\gamma n_1^2]+\omega_0^2n_1=A\omega_0^2\sin\omega t \tag{7}$$

which is just the Van der Pol[3,4,8,9] type of equation. Here a trial solution of the form $n_1=a\times\sin\omega_0t+b\sin(\omega t-\psi)$ is adopted, where both a and b are functions of time, but such that $\omega_0a\gg da/dt$ and $\omega b\gg db/dt$, and that d^2a/dt^2 and d^2b/dt^2 are negligible. Higher harmonics and sum and difference terms are neglected, and so substituting into Eq. (7) the following equations are obtained:

$$3\gamma a^3=2a(2\alpha-3\gamma b^2), \tag{8}$$

$$b(\omega_0^2-\omega_1^2)=A\omega_0^2\cos\psi, \tag{9}$$

$$3\gamma(b^3+2ba^2)\omega-4\alpha b\omega=4A\omega_0^2\sin\psi. \tag{10}$$

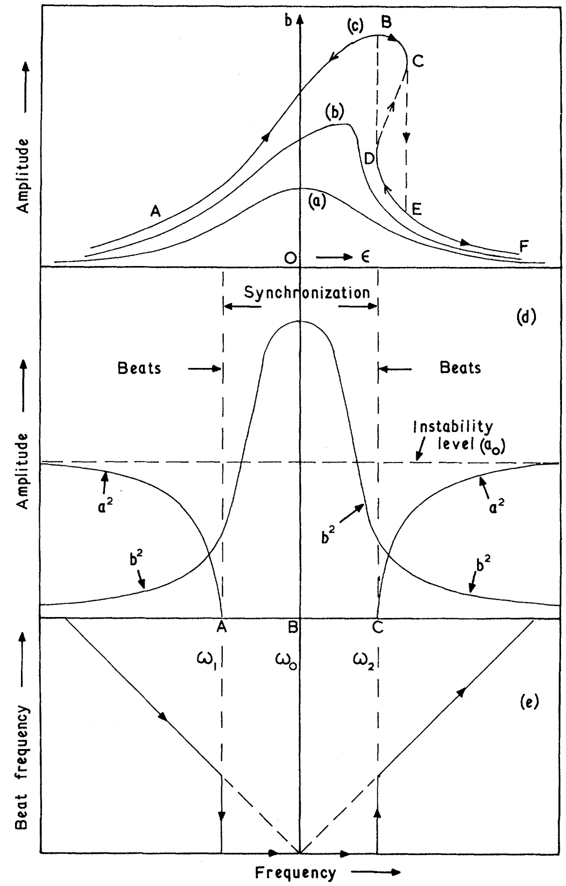

FIG. 1. Theoretical predictions for the behavior of the plasma instability. See text for explanation of (a), (b), (c), (d), and (e).

From Eq. (8) it is seen that in the absence of the driving force the amplitude of the force vibration is given by

$$a_0^2=4\alpha/3\gamma. \tag{11}$$

If, however, the impressed driving force is present and of sufficiently high amplitude that b^2 increases to a value such that the coefficient of a in Eq. (8) is negative, the self-excited oscillation of frequency ω_0 is suppressed. This takes place when

$$b^2=\tfrac{1}{2}a_0^2=2\alpha/3\gamma. \tag{12}$$

Figure 1(d) shows the relationship between a, the amplitude of the free vibration; b, the amplitude of the forced vibration; and the applied frequency ω. The limits of frequency $A\to C$ show the region over which synchronization between the impressed frequency and the internal oscillation occur. Outside this region "beats" are apparent. Figure 1(e) shows how the "beat" frequency should vary with applied frequency ω. The extent of this region of synchronization may be calculated by considering Eqs. (9) and (10) for the conditions when the free vibration is suppressed:

$$b^2(\omega_0^2-\omega^2)+\omega^2b^2(\tfrac{3}{4}\gamma b^2-\alpha)^2=A^2\omega_0^4; \qquad (13)$$

and using the value of b at which synchronization starts and stops from Eq. (12), the synchronization region $\Delta\omega$ is given by

$$\Delta\omega=2^{-1/2}A\omega_0/a_0. \qquad (14)$$

Thus the region of synchronization, or "silent period," is proportional to the driving amplitude A.

In the same way as in case (a), if we apply a signal at a frequency near $\frac{1}{2}\omega_0$ of amplitude proportional to $B\sin\frac{1}{2}\omega_0 t$ it is necessary to replace A by a term $8\beta B^2/9\omega_0^2$ in Eqs. (9) and (10), and in this case the "silent period" $\Delta\omega$ is proportional to B^2.

Experimental.—The experiment was performed in the positive column of a neon arc discharge with a mercury-pool cathode. The arc was run at a constant current of 2 A in an axial magnetic field of 200 G, and this gave a peak density $n_0\sim 3\times10^{11}$ cm^{-3} and a constant temperature $T_e=7.0$ eV. The nature of the instability was determined by using radially moving probes, either to measure floating-potential perturbations, or ion biased in order to measure density oscillations. Axially moving photodiodes were employed to determine the axial wavelength. The instability was found to have predominantly a single frequency independent of magnetic field of 7.5 kHz with an $m=0$ azimuthal mode number and an axial wavelength $\lambda=80$ cm. This corresponds to a velocity of 6.0×10^5 cm, compared with the ion-sound velocity $c_s=(kT_e/M_i)^{1/2}=5.8\times10^5$ cm/sec. Therefore, since the instability appeared to be independent of axial magnetic field and peak density, it was identified as an ion-sound instability.

Externally applied signals were coupled to the plasma by four magnetic coils spaced azimuthally at equal intervals around the discharge column. The plane of each coil was such that they produced an in-phase azimuthal oscillating magnetic field $\vec{B}_\theta^{\sim}$ in the plasma, which by virtue of the $[\vec{B}_\theta^{\sim}\times\vec{E}_r]$ drift produced an oscillating axial velocity v_z in the plasma (here E_r is the zero-order radial electric field in the plasma). By varying the drive current I_0 to these coils the amplitude of the induced velocity (or density) perturbation in the plasma could be changed.

Case (a): instability marginal.—By increasing the neutral pressure in the discharge the ion-neutral collision frequency ν was increased such that the instability was just damped out. Then by using a low driving current ($I_0=3$ A peak to peak) near the frequency $\frac{1}{2}\omega_0$ the curve for the amplitude of the instability shown in Fig. 2(a) was obtained. At a drive current of 4 A the curve shown in Fig. 2(b) was obtained, and it is seen that the resonance curve is now asymmetrical with the maximum shifted to a higher frequency. When the drive is increased to 5 A peak to peak, the curve shown in Fig. 2(c) was obtained, and it is seen that as the frequency is gradually increased the path indicated by the open circles is followed, whereas when the frequency is decreased the path indicated by the crosses is traversed. Therefore, a hysteresis effect is found as indicated by the theoretical pre-

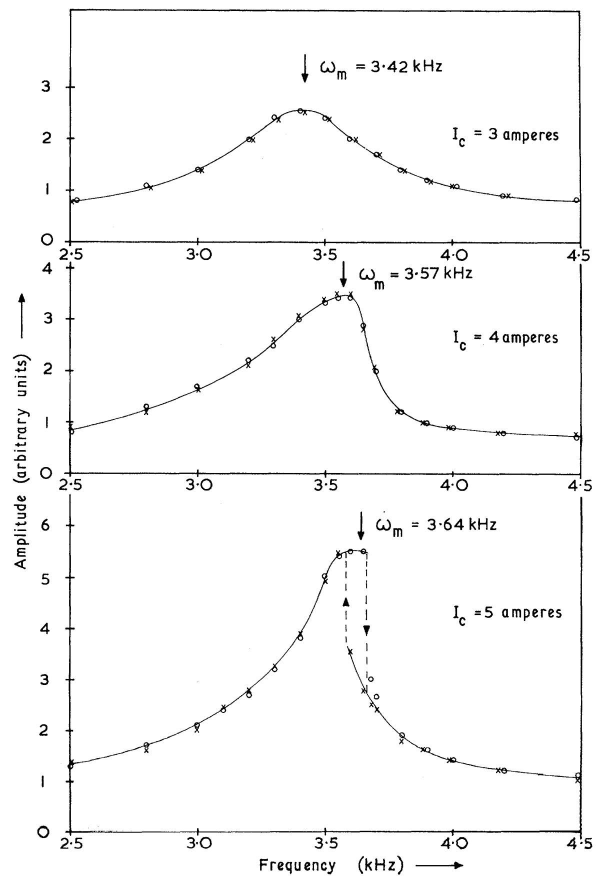

FIG. 2. Experimental amplitude of instability for drive frequency near $\frac{1}{2}\omega_0$. Coil drive currents I_c are the following: (a) 3 A peak to peak, (b) 4 A peak to peak, and (c) 5 A peak to peak.

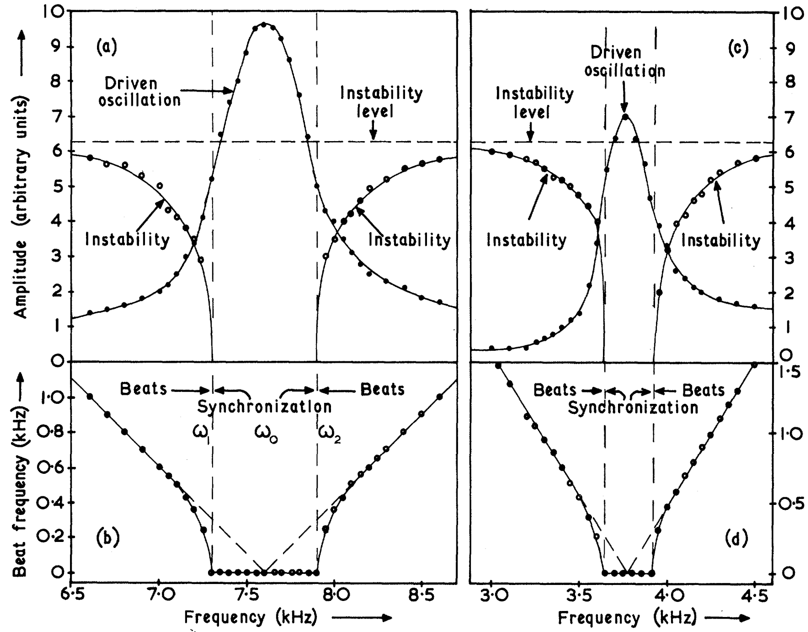

FIG. 3. Experimental behavior of (a) the instability amplitude and the drive amplitude in the plasma, and (b) the "beat" frequency, for a drive frequency near ω_0. (c) Drive amplitude and instability amplitude. (d) Beat frequency for the subharmonic drive near $\frac{1}{2}\omega_0$.

dictions shown in Fig. 1(c). It was also checked that the height of the maximum was proportional to the square of the input current to the coils (I_0^2) as indicated by theory.

Similar hysteresis effects were observed when driving near the fundamental frequency ω_0. The resonance at ω_0 was also induced when driving at $\frac{1}{3}\omega_0$, $\frac{1}{4}\omega_0$, and $\frac{1}{5}\omega_0$.

Case (b): instability present.—As the neutral pressure was decreased the instability reappeared as a self-excited oscillation. In this case, using a constant drive current of 5 A peak to peak the frequency was varied through the resonant frequency of the system. The output from an ion-biased probe in the plasma was fed to a spectrum analyzer, and this allowed the amplitude of the instability, the amplitude of the drive frequency in the plasma, and the beat frequency to be measured simultaneously. Figure 3(a) shows the amplitude of the instability and the drive amplitude obtained by this method, and Fig. 3(b) shows the beat frequency obtained. It is clearly seen that as the frequency increases the instability amplitude decreases and the beat frequency decreases linearly until the frequency ω_1 is reached. At this frequency value the instability is suppressed and frequency synchronization occurs between the driven plasma oscillation and the applied frequency. Above this frequency the driven amplitude in the plasma increases and reaches a maximum at ω_0. At ω_2 the synchronization disappears and the instability reappears together with "beats" of the drive and instability frequencies. This appears to be in complete agreement with the general theoretical predictions as shown in Figs. 1(d) and 1(e). Figures 3(c) and 3(d) show the effect of driving near $\frac{1}{2}\omega_0$ and here similar effects are observed.

Consequently, it is seen that good agreement is obtained between theory and experiment for a nonlinear behavior as specified by Eqs. (4) and (7) and that, in principle, this suggests a method of obtaining values for the particular nonlinear saturation coefficients β and γ relevant to a particular instability in a plasma.

[1] A. Y. Wong and F. Hai, Phys. Rev. Letters 23, 163 (1969).

[2] T. H. Stix, Phys. Fluids 12, 627 (1969).

[3] H. Lashinsky, in Proceedings of the Seventh International Conference on Phenomenon on Ionized Gases, Belgrade, 1965, edited by B..Perovic (Gradevinska Knjiga Publishing House, Belgrade, Yugoslavia, 1966), p. 710.

[4] R. H. Abrams, E. J. Yadlowsky, and H. Lashinsky, Phys. Rev. Letters 22, 275 (1969).

[5] H. G. S. Hsuan, Phys. Rev. 172, 137 (1968).

[6] T. H. Dupree, Phys. Fluids 11, 2680 (1968).

[7] B. Van der Pol, Phil. Mag. 43, 700 (1922).

[8] N. Minorsky, Nonlinear Oscillations (D. Van Nostrand Company, Inc., Princeton, N. J., 1962).

[9] E. V. Appleton, Proc. Cambridge Phil. Soc. 21, 231 (1922).