148, 1380 (1966).
[11]R. Wilson, Phys. Today 22, No. 1, 47 (1969).
[12]T. Massam and A. Zichichi, Nuovo Cimento 43A, 1137 (1966).
[13]M. Conversi et al., Nuovo Cimento 40A, 690 (1965).
[14]D. L. Hartill et al., California Institute of Technology Report No. CALT-68-175 (unpublished).
[15]H. E. Bergeson, J. W. Keuffel, M. O. Larson, G. W. Mason, and J. L. Osborne, Phys. Rev. Letters 21, 1089 (1968).

BARYON OCTET MASSES IN BROKEN SU(3) ⊗ SU(3) *

Yuk-Ming P. Lam and Yee Yen Lee†
The Enrico Fermi Institute, The University of Chicago, Chicago, Illinois 60637
(Received 11 July 1969)

We calculate the contribution to the baryon octet masses of the symmetry breaking term u_0+cu_8 by means of a phenomenological Lagrangian. Our result gives a theoretical justification of the conclusion of von Hippel and Kim.

The success of chiral SU(3) ⊗ SU(3) is based on Nambu's idea[1] that, in the limit of exact symmetry, the pseudoscalar octet has exactly zero masses. Gell-Mann's original idea[2] that this group is broken by a term in the Hamiltonian proportional to $(-u_0-cu_8)$ [where u_i, v_i, for $i=0, 1, \cdots, 8$, form a basis of the representation $(\underline{3}, \underline{3}^*) \oplus (\underline{3}^*, \underline{3})$ of the chiral SU(3) ⊗ SU(3) group, and c is a constant] was recently revised by Gell-Mann, Oakes, and Renner,[3] and the value $c=-1.25$ was suggested. This shows that SU(2) ⊗ SU(2) is a very nearly exact symmetry (and so the pion mass is nearly zero). By using the soft-pion theorem, coupled with a method of extrapolation to the mass shell, von Hippel and Kim[4] put these ideas to work in the domain of threshold amplitudes; and by fitting to experimental data, they obtained the conclusion that the expectation value of u_0 between the octet baryon states,

$$\mu_0 = -\langle B|u_0|B\rangle,$$

is approximately 175 MeV. A recent private communication with von Hippel revealed that he has updated this value to about 200 MeV. As a result, the common mass of the baryon octet before the symmetry was broken was very close to (and slightly lower than) the actual nucleon mass, where the SU(3) ⊗ SU(3) symmetry is already broken. Thus, this result seems to suggest that breaking of SU(3) ⊗ SU(3) only gives the nucleon mass a very small positive increment. In this note, we are going to justify this result by employing the same ideas and a very simple and physically attractive assumption.

We will use the mathematical technique of the phenomenological Lagrangian developed by Callan, Coleman, Wess, and Zumino.[5] The linear subgroup used here is the usual SU(3). The system which we will now consider consists of eight pseudoscalar meson fields ξ_i (which are dimensionless for the moment; actually they are normalized by the factor $F_\pi=F_K=F_\eta$) and eight $\frac{1}{2}^+$ baryons B_i. The (anti-Hermitean) generators V_i, A_i $(i=1,\cdots,8)$ of SU(3) ⊗ SU(3) satisfy the commutation relations

$$[V_i, V_j]=[A_i, A_j]=f_{ijk}V_k,$$

$$[V_i, A_j]=f_{ijk}A_k.$$

For any octuple of fields ξ_i, and any element g of SU(3) ⊗ SU(3), we define the quantity ξ_i' and μ_i' by[6]

$$ge^{\xi\cdot A}=e^{\xi'\cdot A}e^{\mu'\cdot V}. \tag{1}$$

Further, we define

$$B_i'=(e^{\mu'\cdot T})_{ij}B_j, \tag{2}$$

where $(T_i)_{jk}=-f_{ijk}$ are the generators of the eight-dimensional representation of SU(3). The nonlinear transformation of the vector (ξ_i, B_j) under g is given in Ref. 5 as

$$g\colon\ (\xi_i, B_j)\to(\xi_i', B_j'). \tag{3}$$

Then we may interprete the ξ's and B's as the baryon and meson field operators, respectively.

The result of Ref. 5 will now be used to construct functions of ξ_i and B_i that transform linearly under SU(3) ⊗ SU(3). These functions and their transformation properties under SU(3)

$\otimes$ SU(3) are

$$Z^{i;}{}_{j}(\xi,B)=U(\xi)_{m}{}^{i}U(\xi)_{j}{}^{n}B_{n}{}^{m},\quad (\underline{3},\underline{3}^{*}); \tag{4}$$

$$Y_{j;}{}^{i}(\xi,B)=U(-\xi)_{m}{}^{i}U(-\xi)_{j}{}^{n}B_{n}{}^{m},\quad (\underline{3}^{*},\underline{3}); \tag{5}$$

$$G^{i;jk}(\xi,B)=U(\xi)_{r}{}^{i}U(-\xi)_{s}{}^{j}U(-\xi)_{t}{}^{k}(\epsilon^{rsl}B_{l}{}^{t}+\epsilon^{rtl}B_{l}{}^{s}),\quad (\underline{3},\underline{6}); \tag{6}$$

$$H^{jk;i}(\xi,B)=U(-\xi)_{r}{}^{i}U(\xi)_{s}{}^{j}U(\xi)_{t}{}^{k}(\epsilon^{rsl}B_{l}{}^{t}+\epsilon^{rtl}B_{l}{}^{s}),\quad (\underline{6},\underline{3}), \tag{7}$$

where $B_n{}^m$ is the 3×3 traceless baryon matrix, $G^{i;jk}(\xi,B)$ and $H^{jk;i}(\xi,B)$ are symmetric under the interchange of the indices j and k, and

$$U(\xi)=\exp(-i\xi\cdot\lambda/2).$$

Under parity transformation, $Z^{i;}{}_{j}\leftrightarrow Y_{j;}{}^{i}$, $G^{i;jk}\leftrightarrow H^{jk;i}$. Therefore (Y,Z) forms a representation of SU(3) $\otimes$ SU(3) $\otimes$ parity, and so does (G,H). There may be other functions that also transform linearly, but in an argument given later, we will use these two representations only.

The phenomenological Lagrangian consists of two parts: a symmetry-breaking term u_0+cu_8, and an SU(3) $\otimes$ SU(3) symmetric part $\mathcal{L}_0$ which contains the covariant derivatives of the pseudoscalar mesons and baryons, as well as a common mass term for the baryons,[7] and which has already been given in Ref. 5. As for u_0+cu_8, it clearly is a function of the fields ξ_i and B_j, and it may be a linear combination of more than one set of functions, each set transforming like $(\underline{3},\underline{3}^*)\oplus(\underline{3}^*,\underline{3})$. Here we make the assumption that such a combination contains two parts: One is constructed as the bilinear function of the $(\underline{3},\underline{3}^*)\oplus(\underline{3}^*,\underline{3})$ given by (4) and (5) and its conjugate, the other is constructed as the bilinear function of $(\underline{6},\underline{3})\oplus(\underline{3},\underline{6})$ given by (6) and (7), and its conjugate. We assume no other representations enter into this construction.[8]

This assumption may be made plausible by the following consideration. The quarks which form a triplet under the linear SU(3) subgroup can be similarly linearized (with the help of the pseudoscalar mesons) to the $(\underline{1},\underline{3})\oplus(\underline{3},\underline{1})$ representation of SU(3) $\otimes$ SU(3). The triple Kronecker product of this representation with itself yields the Clebsch-Gordan series:

$$[(\underline{1},\underline{3})\oplus(\underline{3},\underline{1})]^3=3[(\underline{3},\underline{3}^*)\oplus(\underline{3}^*,\underline{3})]\oplus3[(\underline{3},\underline{6})\oplus(\underline{6},\underline{3})]\oplus2(\underline{1},\underline{1})\oplus2[(\underline{1},\underline{8})\oplus(\underline{8},\underline{1})]\oplus[(\underline{1},\underline{10})\oplus(\underline{10},\underline{1})]. \tag{8}$$

In the resulting decomposition, only $(\underline{3},\underline{3}^*)\oplus(\underline{3}^*,\underline{3})$ and $(\underline{6},\underline{3})\oplus(\underline{3},\underline{6})$ will give $(\underline{3},\underline{3}^*)\oplus(\underline{3}^*,\underline{3})$ when combined with their respective Dirac conjugates. Notice that other low-dimensional representations such as $(\underline{6}^*,\underline{3}^*)\oplus(\underline{3}^*,\underline{6}^*)$ do not appear in this Clebsch-Gordan series.[9] Since the baryons transform like the combination of three quarks under SU(3), the above considerations guide us to assume that only the $(\underline{3},\underline{3}^*)\oplus(\underline{3}^*,\underline{3})$ and $(\underline{6},\underline{3})\oplus(\underline{3},\underline{6})$ representations are relevant here.

Now we proceed to construct u_i and v_i:

(i) Noticing that $\underline{3}\times\underline{3}=\underline{3}^*\oplus\underline{6}$, we take the conjugate spinor $\bar{Z}_{l;}{}^{j}$ to $Z^{l;}{}_{j}$, multiply it into $Y_{s;}{}^{t}$, and then extract the $(\underline{3},\underline{3}^*)$ representation from this product. With the roles of $Z^{i;}{}_{j}$ and $Y_{j;}{}^{i}$ interchanged, we obtain the $(\underline{3}^*,\underline{3})$ representation. In terms of the usual notation $u_i'(\xi,B)$, $v_i'(\xi,B)$, $i=0,1,\cdots,8$,

$$\begin{pmatrix}u_i'(\xi,B)\\ v_i'(\xi,B)\end{pmatrix}=\left[\exp\begin{pmatrix}0 & -\xi\cdot d\\ \xi\cdot d & 0\end{pmatrix}\right]\begin{pmatrix}u_i'(0,B)\\ v_i'(0,B)\end{pmatrix}, \tag{9}$$

where $(\xi\cdot d)_{jk}=\xi_i d_{ijk}$, and

$$u_i'(0,B)\equiv 2d_{ijk}\bar{B}_jB_k,\quad i,j,k=1,\cdots,8,$$

$$u_0'(0,B)\equiv-(\tfrac{2}{3})^{1/2}\bar{B}\cdot B, \tag{10}$$

$$v_i'(0,B)\equiv0,\quad i=0,1,\cdots,8. \tag{11}$$

Thus (u',v') is obtained from the Kronecker product $(\bar{Y},\bar{Z})\otimes(Y,Z)$.

(ii) Similarly, we note that $\underline{3}^*\times\underline{6}=\underline{3}\oplus\underline{15}$, etc., and so we may extract from the direct product $(\bar{G},\bar{H})$ $\otimes$ (G,H) the $(\underline{3},\underline{3}^*)\oplus(\underline{3}^*,\underline{3})$ portion $[u_i''(\xi,B),\ v_i''(\xi,B)]$ given by relations exactly like (9) and (11), but with (10) replaced by

$$u_i''(0,B)\equiv(4if_{ijk}-6d_{ijk})\bar{B}_jB_k,\quad i=1,\cdots,8.$$

$$u_0''(0,B)\equiv-(6)^{1/2}\bar{B}\cdot B. \tag{12}$$

The linear combination of these two terms then gives the required contribution to u_i and v_i. In general, we have[10]

$$\begin{pmatrix} u_i(\xi,B) \\ v_i(\xi,B) \end{pmatrix} = \left[\exp\begin{pmatrix} 0 & -\xi\cdot d \\ \xi\cdot d & 0 \end{pmatrix}\right]\begin{pmatrix} u_i(0,B) \\ v_i(0,B) \end{pmatrix}, \tag{13}$$

with

$$u_i(0,B) = [iFf_{ijk} + Dd_{ijk} + (\tfrac{3}{2})^{1/2} S\delta_{i0}\delta_{jk}]\bar{B}_j B_k,$$
$$v_i(0,B) = 0, \tag{14}$$

$i = 0, 1, \cdots, 8$, $j, k = 1, \cdots, 8$, where F, D, and S are constants. Yet by inspecting (10) and (12), we arrive at the conclusion that

$$F + D + S = 0. \tag{15}$$

Notice that this result holds for both (10) and (12) separately. Thus, the term $u_0 + cu_8$ in the Lagrangian has a leading term

$$-(\tfrac{3}{2})^{1/2}\{\bar{N}N(1+2^{-1/2}c)(F+\tfrac{1}{3}D) + \bar{\Xi}\Xi[F(1-2^{-1/2}c) + \tfrac{1}{3}D(1+2^{-1/2}c)]$$
$$+ \bar{\Sigma}\Sigma[F + \tfrac{1}{3}D(1-\sqrt{2}c)] + \bar{\Lambda}\Lambda[F + \tfrac{1}{3}D(1+\sqrt{2}c)]\}. \tag{16}$$

When $c \to -\sqrt{2}$, the coefficient of $\bar{N}N$ in (16) vanishes. This justifies the conclusion implied by the work of von Hippel and Kim.

The constants F and D can be fixed once and for all by the actual baryon masses to yield $F \cong 171$ MeV, $D \cong 40$ MeV. The value μ_0 thus calculated is 225 MeV, which agrees with von Hippel's results.

Further observations can be made that the next leading terms in $u_0 + cu_8$ just give the well-known 0 terms in the pseudoscalar-baryon scattering amplitude:

$$u_0 + cu_8 = -[(\tfrac{3}{2})^{1/2}F + 6^{-1/2}D]\bar{B}\cdot B + c(iFf_{ijk} + Dd_{ijk})\bar{B}_j B_k - \tfrac{1}{2}\sum_{i=0}^{8}[(\xi\cdot d)_{0i}^2 + c(\xi\cdot d)_{8i}^2]$$
$$\times[iFf_{ijk} + Dd_{ijk} - (\tfrac{3}{2})^{1/2}(F+D)\delta_{i0}\delta_{jk}]\bar{B}_j B_k + \cdots. \tag{17}$$

We wish to thank Professor Y. Nambu and Professor J. J. Sakurai for valuable discussions, and the former for a critical reading of the manuscript. One of us (Y.Y.L.) is indebted to Professor Y. Nambu and Professor J. J. Sakurai for their hospitality at the Enrico Fermi Institute, and also to the National Science Council of the Republic of China for financial support.

*Work supported in part by the U. S. Atomic Energy Commission under Contract No. AT(11-1)-264.

†On leave of absence from Institute of Physics, National Tsing Hua University, Hsin Chu, Taiwan, Republic of China.

[1]Y. Nambu, Phys. Rev. Letters 4, 380 (1960); Y. Nambu and D. Lurié, Phys. Rev. 125, 1429 (1962), and references therein.

[2]M. Gell-Mann, Phys. Rev. 125, 1067 (1962), and Physics 1, 63 (1964).

[3]M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968).

[4]F. von Hippel and J. K. Kim, Phys. Rev. Letters 22, 740 (1969).

[5]S. Coleman, J. Wess, and B. Zumino, Phys. Rev. 177, 2239 (1969); C. G. Callan, S. Coleman, J. Wess, and B. Zumino, Phys. Rev. 177, 2247 (1969).

[6]The μ's here stand for the u's of Ref. 5.

[7]There may also be a nonderivative coupling term which is invariant under SU(3)⊗ SU(3), but it does not concern us here.

[8]The fruitfulness of this assumption suggests that the quantities given by (4) to (7) may have some significance as items that occur in the Lagrangian. This significance is worth further investigation. An alternative assumption which will give essentially the same end result is that we may construct u_i and v_i as combinations of bilinear functions of $(\underline{3},\underline{3}^*) \oplus (\underline{3}^*,\underline{3})$, $(\underline{6},\underline{3}) \oplus (\underline{3},\underline{6})$, $(\underline{1},\underline{8}) \oplus (\underline{8},\underline{1})$ and their conjugates (indluding all crossed terms) as also suggested by the Clebsch-Gordan series (8). Yet, it should be noticed that $[(\underline{1},\underline{8}) \oplus (\underline{8},\underline{1})] \otimes [(\underline{3},\underline{3}^*) \oplus (\underline{3}^*,\underline{3})]$ reduces to $(\underline{3},\underline{3}^*) \oplus (\underline{3}^*,\underline{3})$ twice, and the assumption further stipulates that only an appropriate one is used to construct $u_0 + cu_8$.

[9]We are using only the transformation properties of the quarks as a guide to our assumption. This is certainly more justifiable than taking quarks as real objects.

[10]We calculate that portion of $u_0 + cu_8$ which contains octet baryons; there may be other terms in $u_0 + cu_8$, for example, those terms that break the mass of the pseudoscalar mesons.