states. The high-quantum-yield pair transitions terminated at $Y(Er^{3+})$ and $^3H_4(Tm^{3+})$, where nonresonant sensitization resulted from the greater spectral density of coupled electronic states relative to the density of coupled ion-phonon states. The collective electronic states and dispersion assigned to exchange were then at least necessary supplements to the ion-pair states of Dexter's[3] theory. Furthermore, the density of excitation states was reduced in the infrared region by the density of pair states formed by the number of possible combinations of ion levels. Therefore, the radiationless relaxation of higher energy levels uncoupled to $Y(Ho^{3+})$, and the weak Er^{3+}- and Tm^{3+}-satellite fluorescence, conceptually were consistent with efficient sensitization.

Further explanation of the experimental results presently would require expanded assumptions. Crystals of varying HoF_3 and TmF_3 concentration are currently being studied.

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AVERAGING METHODS IN NUCLEAR REACTION THEORY*

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It is pointed out that statistical calculations and averages involving overlapping resonances require the use of averaging intervals that are small compared with the energy interval containing the resonance sample. Averages taken over the entire resonance sample or more are physically inappropriate and lead to incorrect relations.

This note is occasioned by a series of recent papers[1-4] in which theoretical averaging methods are employed that do not correspond to physical averages. We shall show that the results of those papers which conflict with earlier work[5] do not correctly represent experimental averages.

Experimental determinations of the energy average $\bar{\sigma}(E_0)$ of a nuclear-resonance cross section $\sigma(E)$ are obtained by procedures that can be represented by

$$\bar{\sigma}(E_0) = \int f(E-E_0, I)\sigma(E)dE, \quad (1)$$

where the resolution function $f(E-E_0, I)$ is normalized to unity and differs from zero appreciably only within an energy interval of length I that is centered on E_0. The precise shape of f depends on the details of the experimental arrangement and is often not known, but the value of f is

presumed to fall off to zero fairly sharply outside the interval I. In any case, the concept of any energy average is useful only if $\bar{\sigma}$ is fairly insensitive to the exact shape of f and is at most a slowly varying function of both I and E_0. This implies that the resonances of σ are not confined to the averaging interval but also occur outside I.

In the case of well-separated resonances, only those resonances which lie inside the interval I contribute to the energy variation of σ in I. Any cumulative contributions from more distant resonances may be approximated by a constant contribution to the background within I. Therefore in this limit the calculation of $\bar{\sigma}(E_0)$ requires theoretical expressions only for the background in I and for those resonances which lie within I. The theoretical resolution function f can be chosen to have the value I^{-1} in the interval I and zero outside (rectangular resolution function).

When resonances overlap, then also some of the resonances which lie outside the interval I will contribute nontrivially to the integral of Eq. (1) because the "tails" of such outside resonances now reach into I and contribute to the energy dependence of the cross section within I. But this effect is confined to the edges of the averaging interval and becomes unimportant at energies that are separated from the edges of I by much more than the resonance widths Γ. It might be thought, therefore, that the correct theoretical average cross section for overlapping resonances could be obtained using a rectangular resolution function f and an expression for σ that contains resonances only within the averaging interval I, provided that I were made sufficiently large compared with Γ. In that way it would appear that the contribution of the edge errors due to the neglect of the outside resonances could be made arbitrarily small.[6]

This supposition is, however, not valid. The averaging procedure outlined in the preceding paragraph is incorrect for overlapping resonances. So are all variants of this procedure which employ theoretical expressions for σ containing resonances only within I, regardless of the precise shape of f.[3,4] The difficulty arises from the requirement of flux conservation (or unitarity of the open-channel S matrix) which imposes certain conditions upon the resonance parameters of σ. For a model having a finite set of N resonances occupying an energy interval Δ the unitarity conditions vary systematically over the interval Δ and the effects of that variation cannot be reduced by increasing the number of resonances, N. As a result one obtains different average formulas depending on whether the averaging interval I is taken equal to Δ as in Refs. 1-4, or is a smaller interval at the center of Δ, as in Ref. 5. The former case is seen to be unphysical, the latter corresponds to the physical case where the resonances in the averaging interval are surrounded by other resonances.

To illustrate the situation we consider the unitary scattering function for a single open channel with N resonances:

$$S = e^{-2i\varphi} \prod_{\mu=1}^{N} \frac{E - E_\mu - \frac{1}{2}i\Gamma_\mu}{E - E_\mu + \frac{1}{2}i\Gamma_\mu}, \tag{2a}$$

which can also be written in the form of a pole expansion

$$S = e^{-2i\varphi}\left(1 - i\sum_{\mu=1}^{N} \frac{G_\mu}{E - E_\mu + \frac{1}{2}i\Gamma_\mu}\right) \tag{2b}$$

which is much easier to average than the manifestly unitary form (2a). In particular, it is well known that the energy average of the sum in Eq. (2b) yields $-i\pi\langle G_\mu\rangle/D$, where $\langle G_\mu\rangle$ is the average of the G_μ and, where D is the spacing of the E_μ in the averaging interval. Similarly the average compound-nucleus cross section is related to the average of the absolute square of the sum in Eq. (2b) which gives $(2\pi/D)\langle |G_\mu|^2/\Gamma_\mu\rangle$ if one assumes that the interference terms average out.[5]

By Eq. (2a) the residues G_μ which we need to calculate averages are not independent parameters but depend on the E_μ and the Γ_μ. The whole question revolves, therefore, upon the comparison of $\langle G_\mu\rangle$ averaged over the resonances in a physical averaging interval $I \ll ND$, on the one hand, and the value of $\langle G_\mu\rangle$ averaged over all N resonances on the other hand.

This question is easily studied with the aid of a simple model of N poles with equal spacings and equal widths[7]:

$$E_{\mu+1} - E_\mu = D, \quad \mu = 1, \cdots, N-1;$$
$$\Gamma_\mu = \Gamma, \quad \mu = 1, \cdots, N. \tag{3}$$

We find from Eq. (2) that

$$\frac{G_\mu}{\Gamma} = \prod_{\substack{k=1-\mu \\ k\neq 0}}^{N-\mu} \left(1 + \frac{i\Gamma}{kD}\right) \tag{4}$$

which immediately leads to the results that in the limit of a large number N of resonances

$$\frac{G_\mu}{\Gamma} \to \frac{\sinh(\pi\Gamma/D)}{\pi\Gamma/D} \text{ when } \mu \approx \tfrac{1}{2}N, \tag{5}$$

$$\frac{G_\mu}{\Gamma} \to \left(\frac{\sinh(\pi\Gamma/D)}{\pi\Gamma/D}\right)^{1/2} \exp\left[\pm i \sum_{k=1}^{N-1} \tan^{-1}\left(\frac{\Gamma}{kD}\right)\right]$$

when $\mu \approx 1$ or N. (6)

Equation (5) has also been shown to be generally valid for the absolute value of the average $\langle G_\mu \rangle$ in a physical averaging interval.[8,9] On the other hand, for the average over all N resonances it can be shown that[2,10]

$$\langle G_\mu \rangle_N = \langle \Gamma_\mu \rangle_N \quad (7)$$

holds rigorously.

We see therefore that for overlapping resonances, that is, for large Γ/D, the relation between the G_μ and the Γ_μ changes radically from Eq. (5) at the center to Eq. (6) at the edge of the interval $\Delta = ND$ and that the average relation Eq. (5) which holds in a physical averaging interval I located near the center is very different from Eq. (7) which gives the average taken over the whole interval $\Delta = ND$.[11]

What are the consequences of using the two kinds of intervals? Using Eq. (2b) one gets for the average[6] over Δ

$$\bar{S}_{I=\Delta} = e^{-2i\varphi}(1-\pi\langle G\rangle_{I=\Delta}/D) = e^{-2i\varphi}(1-\pi\langle\Gamma\rangle/D), \quad (8)$$

and for the optical-model transmission coefficient

$$T_{I=\Delta} = 1-|\bar{S}|^2 = 2\pi(\langle\Gamma\rangle/D)(1-\tfrac{1}{2}\pi\langle\Gamma\rangle/D). \quad (9)$$

On the other hand, averaging over a correct physical averaging interval $I \ll \Delta$ we get from Eq. (5) or Refs. 8 and 9

$$\bar{S}_{I\ll\Delta} = e^{-2i\varphi}[\cosh(\pi\langle\Gamma\rangle/D)-\sinh(\pi\langle\Gamma\rangle/D)] = \exp[-2i\varphi-(\pi\langle\Gamma\rangle/D)], \quad (10)$$

$$T_{I\ll\Delta} = 1-\exp(-2\pi\langle\Gamma\rangle/D). \quad (11)$$

The term $\cosh(\pi\langle\Gamma\rangle/D)$ in Eq. (10) comes from the contribution of the resonances outside I to the background of S within the physical averaging interval I.[12]

The average compound-nucleus cross section for $I=\Delta$ has been evaluated—albeit incorrectly[13]—in Ref. 4. For the correct physical case $I \ll \Delta$ it has been shown in Ref. 5 that the fluctuation cross section can be written in the form

$$\sigma_{cc'}^{(\mathrm{fl})} = (\pi/k_c^2)(\langle\Theta_{\mu c}\Theta_{\mu c'}/\Theta_\mu\rangle - M_{cc'}), \quad (12)$$

where $\Theta_\mu = \sum_c \Theta_{\mu c}$ and the quantities $M_{cc'}$ and the averages $\langle\Theta_{\mu c}\rangle$ depend not only upon the channel transmission coefficients but in general also upon the details of the distributions and correlations of resonance parameters. However, in the limit where all channel transmission coefficients are small,[5]

$$\Theta_{\mu c} \to 2\pi\Gamma_{\mu c}/D, \quad M_{cc'} \to 0, \quad (13)$$

which yields Bethe's average cross-section formula[14] with width-fluctuation correction factor.[15]

On the other hand, in the limit of large Γ/D under the usual statistical assumption of uncorrelated channels, it was found that[5]

$$\langle\Theta_{\mu c}\rangle \to T_c, \quad M_{cc'} \to 0, \quad (14)$$

resulting in the familiar Hauser-Feshbach formula[16] again with a Θ fluctuation correction where necessary.

In the domain of intermediate conditions where neither (11) nor (10) is applicable, the average cross-section calculation is more complicated and requires information on the statistical distributions and correlations of the resonance parameters E_μ, Γ_μ, and $G_{\mu c}$. This information, which is also required for interpretation of cross-section fluctuations, has been discussed elsewhere.[5,7]

We conclude that resonances outside the averaging interval have a large effect upon the calculated averages over overlapping resonances and that this effect cannot be ignored. The averaging interval I must be surrounded by another interval Δ sufficiently large so that only resonances outside Δ contribute a negligible energy dependence to the S matrix within I. The background and resonance parameters within Δ must be chosen according to an appropriate physical model. Resonances outside Δ can either be omitted or they can be chosen at pleasure, provided they yield a convergent constant contribution within I that can be compensated by an adjustment of the background term of the S matrix. This is the basis for the very convenient "statistical S matrix" which has a statistically uniform distribution of resonances from $E = -\infty$ to $+\infty$ and a constant background. The definition and use of this statistical S matrix is discussed in detail in Ref. 5. Effects of energy-variable statistics were discussed in Ref. 8.

Besides the rectangular resolution function the Lorentzian resolution function[17]

$$f(E,E_0) = \frac{I/2\pi}{(E-E_0)^2+\frac{1}{4}I^2} \quad (15)$$

has been used extensively in theoretical calculations. Of course its use very obviously requires

a device such as the statistical S matrix. Otherwise the long tails of the Lorentzian will pick up contributions that are not typical of the resonances within the more sharply defined experimental averaging interval I.[18] When this is done the Lorentzian yields results that are identical to those obtained with a rectangular resolution function.

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STIMULATED THERMAL SCATTERING OF SHORT LIGHT PULSES

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The theory of stimulated thermal scattering of trains of short light pulses is developed. Peculiar new effects are predicted for forward scattering, especially a critical dependence on the angle between scattered light and pump light. Particular emphasis is directed to the problem of pure thermal scattering. The dominant contribution to the stimulated scattering is in general due to density variations, no matter how short the light pulses are. Only under specific conditions is the density variation eliminated, allowing the observation of the pure thermal effect.

Considerable interest has been focused recently on the problem of stimulated scattering (STS)[1,2] of mode-locked laser pulses.[3-5] The amplification of a beam of weak pulses passing through an absorbing liquid in the presence of a strong beam has been reported by Mack.[3,4] It was suggested by Batra and Enns[5] that second sound could also be detected in this way.

In view of these important aspects, a theoretical investigation of STS of mode-locked light pulses seems to be necessary. The small-signal and amplification technique developed previously[6-8] is here extended to trains of short light pulses.

The results provide a simple explanation of some of the experimental results and, most important, allow one to distinguish between STS and potentially superimposed effects. In short, we find a critical influence of the Brillouin frequency ω_B and of the spacing between subsequent pulses, t_S. The amplification of the scattered wave is