ing the same phase for the unitarity sum of S-matrix decay amplitudes.

The remaining constraints in Eq. (7), combined with Eq. (15), impose many additional conditions on the decay amplitudes (especially for $K_L \to 2\pi$), which are discussed in the succeeding Letter.[7]

It is a pleasure to thank L. Durand, III, C. J. Goebel, and B. Sakita for their solicitous interest and stimulating conversations.

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[1]J. S. Bell and J. Steinberger, in Proceedings of the Oxford International Conference on Elementary Particles, September, 1965 (Rutherford High Energy Laboratory, Chilton, Berkshire, England, 1966), pp. 195-222.

[2]R. C. Casella, Phys. Rev. Letters 21, 1128 (1968).

[3]W. D. McGlinn and D. Polis, Phys. Rev. Letters 22, 908 (1969).

[4]We note that since the total widths have been factored out of the pole terms in Eq. (2), the g's and h's are dimensionless.

[5]K. W. McVoy, to be published.

[6]This is true only to lowest order in Γ_L/Γ_S. It has long been known [e.g., A. M. Lane and R. G. Thomas, Rev. Mod. Phys. 30, 257 (1958)] that in general the sum of the partial widths exceeds the total width, for each resonance.

[7]L. Durand, III, and K. W. McVoy, following Letter [Phys. Rev. Letters 23, 59 (1969)].

S-MATRIX DESCRIPTION OF K_L AND K_S DECAYS*

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We show how the usual phenomenological description of the decays of the K_S and K_L mesons can be derived in a unified manner beginning from a description of the K_S and K_L states as overlapping resonances in a scattering matrix. The unitarity relations for overlapping resonances in a CPT-invariant (but not CP- or T-invariant) theory play a crucial role in the discussion, and are treated in detail.

In the present paper, we show how the usual phenomenological description of the decays of the neutral K mesons K_S and K_L can be derived in a simple, unified manner beginning from a description of the K_S and K_L states as overlapping resonances in a scattering matrix. The unitarity relations for the CPT-invariant S matrix are found to play a central role in the discussion of all decay modes of K_S and K_L. In the customary analysis,[1] on the other hand, unitarity is used only in the discussion of the CP-nonconserving decays, to determine the phase of the amplitude ratios

$$\epsilon = A(K_L \to \pi\pi,\ I=0)/A(K_S \to \pi\pi,\ I=0) \tag{1}$$

(through the Bell-Steinberger sum rule[1,2]), and

$$\epsilon' = A(K_L \to \pi\pi,\ I=2)/\sqrt{2}A(K_S \to \pi\pi,\ I=0) \tag{2}$$

(through the Watson final-state theorem applied to K and $\bar{K}$ decays). It does not enter the standard discussion of the semileptonic decay modes of K_L and K_S at all.

(a) General formulation.—The K_S and K_L mesons are overlapping resonances which decay into a common set of channels (predominantly, the 2π, 3π, $\pi l\bar{\nu}$, and $\pi\bar{l}\nu$ channels). If the energy dependence of the background scattering in these channels can be neglected in the neighborhood of the K_S and K_L masses, the partial-wave S matrix connecting the relevant channels can be approximated by the two-pole expression

$$S(E) = B - i\Gamma_S \frac{g_S\tilde{h}_S}{E-\xi_S} - i\Gamma_L \frac{g_L\tilde{h}_L}{E-\xi_L}. \tag{3}$$

The constant background matrix B describes that part of the scattering ($2\pi \to 2\pi$, $3\pi \to 3\pi$, etc.) not associated with K_S and K_L. ξ_S and ξ_L are the complex resonance energies for the K_S and K_L systems, $\xi_S = m_S - i\Gamma_S/2$ and $\xi_L = m_L - i\Gamma_L/2$. g_S and g_L are (constant) column vectors of partial-width amplitudes g_{Sc}, g_{Lc} for the decay of K_S and K_L into channel c, and $\tilde{h}_S$ and $\tilde{h}_L$ are the corresponding row vectors which describe the production of K_S and K_L through those channels. The usual decay and production amplitudes are related to the g's and h's by $A(K_S \to c) = \Gamma_S^{1/2}g_{Sc}$, $A(K_L \to c) = \Gamma_L^{1/2}g_{Lc}$, $A(c \to K_S) = \Gamma_S^{1/2}h_{Sc}$, $A(c \to K_L) = \Gamma_L^{1/2}h_{Lc}$.

The requirement that S be unitary throughout the K_S-K_L region leads to a unitarity relation for the background matrix B,

$$BB^\dagger = B^\dagger B = 1, \tag{4}$$

and two vector equations which must be satisfied by the g's and h's,[3]

$$Bh_S^* - i\frac{\Gamma_L}{\xi_S^* - \xi_L}(h_S^\dagger h_L)g_L - (h_S^\dagger h_S)g_S = 0, \quad (5)$$

$$Bh_L^* - i\frac{\Gamma_S}{\xi_L^* - \xi_S}(h_L^\dagger h_S)g_S - (h_L^\dagger h_L)g_L = 0. \quad (6)$$

CPT invariance leads to additional relations among the g's and h's. These assume a simple form if we choose the channel states to be eigenstates of CP with eigenvalues ± 1. We will write g_S and g_L in a split notation as

$$g_S = \begin{pmatrix} g_S^+ \\ g_S^- \end{pmatrix}, \quad g_L = \begin{pmatrix} g_L^+ \\ g_L^- \end{pmatrix}, \quad (7)$$

where the $\pm$ signs refer to the CP eigenvalues of the channel states. The h's then assume the form[2]

$$h_S = \begin{pmatrix} h_S^+ \\ h_S^- \end{pmatrix} = \begin{pmatrix} g_S^+ \\ -g_S^- \end{pmatrix},$$

$$h_L = \begin{pmatrix} h_L^+ \\ h_L^- \end{pmatrix} = \begin{pmatrix} -g_L^+ \\ g_L^- \end{pmatrix} \quad (CPT). \quad (8)$$

Note in particular that $h_S^\dagger h_S = g_S^\dagger g_S$, $h_L^\dagger h_L = g_L^\dagger g_L$, and $h_S^\dagger h_L = -g_S^\dagger g_L$.[4] These relations and Eq. (8) can be used to express the unitarity equations entirely in terms of the decay amplitudes $g_S^\pm$ and $g_L^\pm$.

CPT invariance requires that the background matrix be of the form

$$B = \begin{pmatrix} B_{++} & B_{+-} \\ B_{-+} & B_{--} \end{pmatrix}, \quad (9)$$

where $\tilde{B}_{++} = B_{++}$, $\tilde{B}_{--} = B_{--}$, and $B_{-+} = -\tilde{B}_{+-}$. We will assume in the ensuing discussion that the background scattering conserves CP, that is, that CP invariance is violated only by the K_S and K_L terms in Eq. (3).[5] In this case, B_{+-} and B_{-+} vanish, while B_{++} and B_{--} are separately unitary and symmetric. It is rather easy to prove, independently of this assumption, that $g_S^\dagger g_S$ is equal to $g_L^\dagger g_L$, and that the quantity $(ig_S^\dagger g_L)/(\xi_S^* - \xi_L)$ is real[6]:

$$ig_S^\dagger g_L/(\xi_S^* - \xi_L) = \alpha, \quad \alpha \text{ real}, \quad (10)$$

$$g_S^\dagger g_S = g_L^\dagger g_L = [1 + \Gamma_S\Gamma_L\alpha^2]^{1/2} \geq 1. \quad (11)$$

The unitarity equations assume a remarkably simple form when we use these consequences of CPT invariance:

$$Bh_S^* = [1 + \Gamma_S\Gamma_L\alpha^2]^{1/2} g_S - \Gamma_L\alpha g_L, \quad (12)$$

$$Bh_L^* = [1 + \Gamma_S\Gamma_L\alpha^2]^{1/2} g_L - \Gamma_S\alpha g_S. \quad (13)$$

Note that α is proportional to the inner product $g_S^\dagger g_L$ which describes the overlap of K_S and K_L in the decay channels. The Bell-Steinberger sum rule[1] for the K_S and K_L decays is equivalent in the case of CPT conservation to the statement that $g_S^\dagger g_L = i(\xi_S^* - \xi_L)\alpha$, with $\alpha = \langle K_L | K_S \rangle$ real. If CP were conserved, with K_S and K_L being CP eigenstates with eigenvalues +1 and −1, g_S^- and g_L^+ would vanish. In this limit, $g_S^\dagger g_L = 0$, α vanishes, and there are no connections between the decays of K_S and K_L. It is interesting to note that the sums of the partial widths $\Gamma_S g_S^\dagger g_S$ and $\Gamma_L g_L^\dagger g_L$ for the decays of K_S and K_L given by Eq. (11) are larger than the total widths Γ_S and Γ_L. This characteristic of overlapping resonances has long been recognized.[7]

(b) Applications.—The general unitarity equations can be simplified significantly if we restrict our attention to the K_S-K_L system. We will use the following information[8]: (i) K_S and K_L are very nearly eigenstates of CP with eigenvalues +1 and −1; (ii) $m_L - m_S \sim \Gamma_S/2$, $\Gamma_S/\Gamma_L \sim 600$; (iii) $|g_{S\to\pi\pi}(I=2)|/|g_{S\to\pi\pi}(I=0)| \sim 0.06$; (iv) the partial decay rates for the semileptonic decays of K_L and K_S and 3π decays of K_L are comparable; (v) the magnitude of the CP-nonconserving 2π decay amplitude of K_L is $\sim 2\times 10^{-3}$ of the CP-conserving 2π decay amplitude of K_S.

We first consider the magnitude of the parameter α, $|\alpha| \simeq \sqrt{2}|g_S^\dagger g_L|/\Gamma_S$. We will assume that the strength of the CP-invariance violation is small in all channels, specifically, that $|\Gamma_L^{1/2} \times g_{Lc}^+| \sim 10^{-3}|\Gamma_S^{1/2} g_{Sc}^+|$, and $|\Gamma_S^{1/2} g_{Sc}^-| \sim 10^{-3} \times |\Gamma_L^{1/2} g_{Lc}^-|$. It is then easily seen that the only important contribution to $g_S^\dagger g_L$ arises from the 2π decays of K_S and K_L in the $I=0$ state.[9] If we retain only this contribution, we obtain an explicit expression for α,

$$\alpha \simeq i g_{S\to 2\pi}^*(I=0) g_{L\to 2\pi}(I=0)/(\xi_S^* - \xi_L). \quad (14)$$

This parameter is quite small experimentally ($\Gamma_S|\alpha| \sim 3\times 10^{-3}$). We will therefore replace the factors $[1 + \Gamma_S\Gamma_L\alpha^2]$ by unity in Eqs. (12) and (13). We will also drop quantities which are second order in the CP nonconservation. The unitarity equations then reduce to the following:

$$B_+ g_S^{+*} = g_S^+, \quad (15)$$

$$B_- g_L^{-*} = g_L^-, \quad (16)$$

$$-B_+ g_L^{+*} = g_L^+ - \Gamma_S\alpha g_S^+, \quad (17)$$

$$-B_- g_S^{-*} = g_S^- - \Gamma_L\alpha g_L^-. \quad (18)$$

These equations lead immediately to the usual parametrization of the CP-nonconserving decays of K_L and K_S.

$K_L\to 2\pi$.—We will assume that the strong-interaction background scattering in the $\pi\pi$ system is diagonal in the isospin. The phases of the CP-allowed 2π decay amplitudes of K_S are then determined by Eq. (15),

$$g_{S\to 2\pi}(I=0)=\pm e^{i\delta_0}|g_{S\to\pi\pi}(I=0)|, \quad (19)$$

$$g_{S\to 2\pi}(I=2)=\pm e^{i\delta_2}|g_{S\to\pi\pi}(I=0)|, \quad (20)$$

where δ_0 and δ_2 are the strong-interaction phase shifts for $\pi\pi$ scattering in the $I=0$ and $I=2$ states. Equations (19) and (20) are of course just the statement of the Watson final-state interaction theorem for the CP-allowed 2π decays of K_S. The dominance of the $I=0$ 2π decay mode of K_S and the normalization condition in Eq. (11) imply in addition that $|g_{S\to\pi\pi}(I=0)|\sim 1$, so that $g_{S\to\pi\pi}(I=0)=\pm e^{i\delta_0}$.

The 2π decay amplitudes for K_L are restricted by Eq. (17),

$$e^{2i\delta_0}g_{L\to\pi\pi}^*(I=0)=-g_{L\to\pi\pi}(I=0)+\Gamma_S\alpha g_{S\to\pi\pi}(I=0), \quad (21)$$

$$e^{2i\delta_2}g_{L\to\pi\pi}^*(I=2)=-g_{L\to\pi\pi}(I=2). \quad (22)$$

We have dropped the small $I=2$ 2π decay amplitude of K_S in writing Eq. (22). Equations (19), (21), and (22) and the explicit expression for α given in Eq. (14) are sufficient to determine the phases of the CP-nonconserving 2π decay amplitudes of K_L,

$$g_{L\to\pi\pi}(I=0)=\pm ie^{i(\delta_0-\Delta)}|g_{L\to\pi\pi}(I=0)|, \quad (23)$$

$$g_{L\to\pi\pi}(I=2)=\pm ie^{i\delta_2}|g_{L\to\pi\pi}(I=2)|, \quad (24)$$

where

$$\Delta=\tan^{-1}\frac{\Gamma_S/2}{m_L-m_S}. \quad (25)$$

It is customary to introduce the amplitude ratios defined in Eqs. (1) and (2), $\epsilon=\Gamma_L^{1/2}g_{L\to\pi\pi}(I=0)/\Gamma_S^{1/2}g_{S\to\pi\pi}(I=0)$, and $\epsilon'=\Gamma_L^{1/2}g_{L\to\pi\pi}(I=2)/[\sqrt{2}\Gamma_S^{1/2}\times g_{S\to\pi\pi}(I=0)]$. The results in Eqs. (23) and (24) determine the phases of ϵ and ϵ',

$$\epsilon=\pm ie^{-i\Delta}|\epsilon|,\ \ \epsilon'=\pm ie^{i(\delta_2-\delta_0)}|\epsilon'|, \quad (26)$$

in agreement with the results of the standard discussions.[1] The amplitude ratios $\eta_{+-}=A(K_L\to\pi^+\pi^-)/A(K_S\to\pi^+\pi^-)$ and $\eta_{00}=A(K_L\to\pi^0\pi^0)/A(K_S\to\pi^0\pi^0)$ are given by $\eta_{+-}=\epsilon+\epsilon'$, $\eta_{00}=\epsilon-2\epsilon'$ as in the Wu-Yang analysis.[1]

Equation (21) can also be written in a form which allows us to relate ϵ and α,

$$\alpha=2\,\mathrm{Re}[g_{L\to\pi\pi}(I=0)/\Gamma_S g_{S\to\pi\pi}(I=0)]=2\,\mathrm{Re}\epsilon/(\Gamma_S\Gamma_L)^{1/2}. \quad (27)$$

The magnitudes of ϵ and ϵ' are of course not determined by the present considerations.

Semileptonic decays.—The background scattering in the leptonic channels $\pi l\nu\to\pi l\nu$, with l an electron or muon, is electromagnetic or weak; the background matrix B for the decay of K_S and K_L into these channels may therefore by taken as a unit matrix. We can then conclude from Eqs. (15) and (16) that the g's for the CP-allowed decays of K_S and K_L are real,

$$g_{S\to\pi l\nu}^+,\ g_{L\to\pi l\nu}^-\ \text{real}. \quad (28)$$

Similarly, Eqs. (17), (18), and (27) determine the real parts of the CP-nonconserving decay amplitudes,

$$\Gamma_L^{1/2}\mathrm{Re}g_{L\to\pi l\nu}^+=\Gamma_S^{1/2}g_{S\to\pi l\nu}^+\mathrm{Re}\epsilon, \quad (29)$$

$$\Gamma_S^{1/2}\mathrm{Re}g_{S\to\pi l\nu}^-=\Gamma_L^{1/2}g_{L\to\pi l\nu}^-\mathrm{Re}\epsilon. \quad (30)$$

The decays of K_L and K_S into states with definite lepton charge are described by linear combinations of the amplitudes for decay into the states with definite CP,

$$A(K_{L,S}\to\pi^\mp l^\pm\nu)=2^{-1/2}[A^+(K_{L,S}\to\pi l\nu)\pm A^-(K_{L,S}\to\pi l\nu)]. \quad (31)$$

We note also the expression of the $\Delta S=\Delta Q$ rule in the present formalism,

$$A^+(K_S\to\pi l\nu)=A^-(K_L\to\pi l\nu),\ \ A^-(K_S\to\pi l\nu)=A^+(K_L\to\pi l\nu). \quad (32)$$

The charge asymmetry in the decays of K_L and K_S is easily calculated using Eqs. (28)-(31). To lowest order in the CP-invariance violation,

$$R_{L,S}=\frac{|A(K_{L,S}\to\pi^-l^+\nu)|^2-|A(K_{L,S}\to\pi^+l^-\bar{\nu})|^2}{|A(K_{L,S}\to\pi^-l^+\nu)|^2+|A(K_{L,S}\to\pi^+l^-\bar{\nu})|^2}=2\,\mathrm{Re}\epsilon. \quad (33)$$

A failure of the $\Delta S=\Delta Q$ rule would introduce an extra factor A_S^+/A_L^- in R_L, and a factor A_L^-/A_S^+ in R_S.

3π decays.—The unitarity relations for the 3π decays of K_L and K_S, Eqs. (16) and (18), involve integrals of the $3\pi\to 3\pi$ background amplitude in B_- and either g_L^{-*} or g_S^{-*} over the 3π phase space (Dalitz plot), and have not been investigated.

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[1]T. T. Wu and C. N. Yang, Phys. Rev. Letters 13, 380 (1964); J. S. Bell and J. Steinberger, in Proceedings of the Oxford International Conference on Elementary Particles, September, 1965 (Rutherford High Energy Laboratory, Chilton, Berkshire, England, 1966), p. 195.

[2]K. W. McVoy, preceding Letter [Phys. Rev. Letters 23, 56 (1969)].

[3]The analogs of Eqs. (5) and (6) for CPT and T-conserving interactions are discussed by K. W. McVoy (to be published). The more general relations in Eqs. (5) and (6) can be derived quite simply from the analytically continued unitarity equation $S^\dagger(E^*)S(E)=S(E)S^\dagger(E^*)=1$ by requiring that the residues of the poles in these expressions at $E=\xi_S$ and $E=\xi_L$ vanish. The inner products in Eqs. (5) and (6) are understood to include sums over spins and integrals over phase space where these are necessary.

[4]Had we assumed T invariance rather than CPT invariance, the g's and h's would be equal, $g_S=h_S$, $g_L=h_L$, and S would be symmetric.

[5]This assumption is justified if CP is conserved by the strong and electromagnetic interactions. CP-invariance violations in the nonstrange weak interactions will change our results only by terms of weak interaction strength (10^{-6}) relative to the g's. The presence of CP-nonconserving terms in the strong or electromagnetic background scattering of order 10^{-3} relative to the CP-conserving terms would necessitate a complete reanalysis of the K_S and K_L decay phenomenology with B given by Eq. (9). Equations (10)-(13) are still valid, and provide a simple starting point for this analysis.

[6]The proof is based on an identity $g_L^\dagger Bh_S^*=-g_S^\dagger Bh_L^*$ which follows from the properties of the g's, h's, and B. Multiply Eq. (5) by $g_L^\dagger$ and Eq. (6) by $g_S^\dagger$, and equate the expressions for $g_L^\dagger Bh_S^*$ and $-g_S^\dagger Bh_L^*$. Upon collecting terms, one finds that the ratio $(g_S^\dagger g_S)/(g_L^\dagger g_L)$ is necessarily equal to unity, and that $(g_S^\dagger g_L)/(\xi_S^*-\xi_L)$ is pure imaginary. The value of $g_S^\dagger g_S$ is established by using Eqs. (5) and (6) to determine the right-hand sides of the identities $g_S^\dagger g_S=\tilde{h}_S B^\dagger Bh_S^*$, $g_L^\dagger g_L=\tilde{h}_L B^\dagger Bh_L^*$, and eliminating the quantity $\mathrm{Re}(g_S^\dagger g_L)$ between the two equations.

[7]A. M. Lane and R. G. Thomas, Rev. Mod. Phys. 30, 257 (1958). One can easily establish a sum rule for the total widths. However, this envolves the production, decay and background amplitudes,

$$\Gamma_S g_S^\dagger Bh_S^* + \Gamma_L g_L^\dagger Bh_L^* = \Gamma_S + \Gamma_L.$$

[8]J. Cronin, in Proceedings of the Fourteenth International Conference on High Energy Physics, Vienna, Austria, September, 1968 (CERN Scientific Information Service, Geneva, Switzerland, 1968), p. 281. Also N. Barash-Schmidt et al., Rev. Mod. Phys. 41, 109 (1969).

[9]With the assumptions noted above, the contribution of the CP-nonconserving 2π decay in the $I=2$ state is suppressed relative to that of the $I=0$ 2π decay by a factor of $\sim\frac{1}{16}$, while the contributions of the CP-nonconserving 3π and semileptonic decay modes of K_S and K_L are suppressed by roughly a factor $\Gamma_L/\Gamma_S\sim 1/25$.

ERRATA

SCATTERING MODEL OF MOLECULAR ELECTRONIC STRUCTURE. Franklin C. Smith, Jr., and Keith H. Johnson [Phys. Rev. Letters 22, 1168 (1969)].

Reference 3 should read: K. H. Johnson, Intern. J. Quantum Chem. 1S, 361 (1967).

EXPERIMENTAL VALUE OF $\Delta E_\mathrm{H}-\mathcal{S}_\mathrm{H}$ IN HYDROGEN. T. W. Shyn, W. L. Williams, R. T. Robiscoe, and T. Rebane [Phys. Rev. Letters 23, 1273 (1969)].

The quoted error on $\Delta E_\mathrm{H}-\mathcal{S}_\mathrm{H}$ is one standard deviation from the mean, not one average deviation as reported.