yield the usual moving trajectories.

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PARTIAL-WIDTH FORMULATION OF UNITARITY SUM RULES FOR K_L-K_S DECAY

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By including K_S in the background which underlies K_L, S-matrix unitarity can be used to derive a set of constraints on the partial decay amplitudes of the two resonances. The only constraint which does not explicitly involve the strong-interaction background phases is equivalent to the Bell-Steinberger sum rule.

In 1965, Bell and Steinberger[1] derived a "unitarity sum rule,"

$$-i(M_L^*-M_S)\langle L|S\rangle=\sum\langle F|T|L\rangle^*\langle F|T|S\rangle, \quad (1)$$

satisfied by the amplitudes for the decay of K_L and K_S into final states F. Its practical significance is that it determines the phase of the right-hand side ($\sum$) in terms of $\langle L|S\rangle$, and $\langle L|S\rangle$ in turn is, e.g., real if CPT is conserved,[1] and imaginary if only T is conserved.[2] The sum rule is particularly noteworthy because it involves only the amplitudes for decay into the channel states F, and not the production amplitudes from the channels (2π, 3π, etc.), which are different if T is not conserved (and unobservable because they describe production via the weak interaction).

Although the Bell-Steinberger derivation involved only a consideration of the time dependence of the decay process, McGlinn and Polis[3] have recently suggested that by regarding K_L and K_S as conventional but overlapping resonances in the channels open at that energy, it should be possible to obtain an equivalent sum rule in terms of K-matrix partial-width amplitudes. They did so, but the sum rule they found in this way appeared to be different from the Bell-Steinberger result. The apparent difference seems to us to arise from a failure to distinguish between K-matrix and S-matrix decay amplitudes. To explain this, we have obtained yet a third sum rule, this one expressed in terms of S-matrix partial widths. We find that the phase information it contains is equivalent both to that of the Bell-Steinberger sum rule and to that of the McGlinn-Polis expression, so that in this sense all three results are equivalent.

Consider two overlapping resonances with the same quantum numbers. If in their neighborhood the energy dependence of the background can be neglected, the partial-wave S matrix for N open channels can be approximated by the two-pole expression

$$S(E)=B-i\Gamma_S\frac{g_S\tilde{h}_S}{E-m_S+\frac{1}{2}i\Gamma_S}-i\Gamma_L\frac{g_L\tilde{h}_L}{E-m_L+\frac{1}{2}i\Gamma_L}, \qquad (2)$$

with B an $N\times N$ constant matrix. g_S and g_L are column vectors of partial-width amplitudes g_{Sc} and g_{Lc} for the decay of K_S and K_L into channels c. $\tilde{h}_S$ and $\tilde{h}_L$ are corresponding row vectors for production from these channels (with $h\neq g$ if T is not conserved), so that $g\tilde{h}$ is an $N\times N$ nonsymmetric dyad, $(g\tilde{h})_{cc'}=g_c h_{c'}$.[4]

We wish to impose two distinct conditions on S: that it be unitary, and that it be CPT invariant. Taking unitarity first, it is clear that S can be identically unitary in E only if the vectors g and h satisfy a number of constraints. The exact constraint equations have been obtained elsewhere,[5] but in the case $\Gamma_L \ll \Gamma_S$ an adequate approximation can be obtained by the following simple argument. First, from the fact that S must be unitary outside of both resonances, it clearly follows that B itself must be unitary. Then if $\Gamma_L\ll\Gamma_S$, there is a large energy region inside the broad K_S but outside the narrow K_L where the sum of the first two terms of Eq. (2) alone must be unitary; requiring this imposes constraints on g_S and h_S. Finally, unitarity within K_L imposes constraints on g_L and h_L as well. In the T-conserving case $g=h$, these constraints turn out to be simply a restatement of the Watson final-state theorem, which determines the phases of the decay amplitudes g_{Sc} and g_{Lc} to be the scattering phases of their respective backgrounds in channel c.

Thus we have, first,

$$B^\dagger B=1. \qquad (3)$$

Secondly, the one-pole expression including only K_S is readily seen to be unitary for real E only if

$$B^\dagger g_S=(g_S{}^\dagger g_S)h_S{}^*. \qquad (4)$$

Squaring the latter equation gives

$$(g_S{}^\dagger g_S)(h_S{}^\dagger h_S)=1, \qquad (5)$$

but since only $g_S\tilde{h}_S$ appears in S, their relative normalizations are immaterial and we can take

$$g_S{}^\dagger g_S=h_S{}^\dagger h_S=1. \qquad (6)$$

In other words, the sum of the decay widths as well as the sum of the production widths equals the total width Γ_S.[6] Equations (4) and (6) ($N=2$ constraints) are the unitarity conditions for K_S alone. We note in passing that if T were conserved, so that $h_S=g_S$, and if B were diagonal, $B_{cc'}=\delta_{cc'}e^{2i\delta_c}$, Eq. (4) would simply determine the phase of g_{Sc} to be δ_c, i.e., the Watson final-state theorem familiar from the customary Breit-Wigner expression for S.

If now the first two terms of Eq. (2) are called B_S and used as the background for K_L, we have as before $(g_L{}^\dagger g_L)=(h_L{}^\dagger h_L)=1$, and $B_S{}^\dagger g_L=h_L{}^*$, which is, explicitly,

$$Bh_L{}^*-i\frac{\Gamma_S}{m_L-m_S+\frac{1}{2}i\Gamma_S}(h_L{}^\dagger h_S)g_S=g_L. \qquad (7)$$

These are the unitarity constraints on the K_L amplitudes, with the K_S component of the K_L background represented by the second term (which would vanish if K_L and K_S decayed only into channels of different CP). Multiplying the equation on the left by $g_S{}^\dagger$ and using Eqs. (4) and (6) produces the equation

$$h_L{}^\dagger h_S=e^{2i\Delta}(g_L{}^\dagger g_S)^*, \qquad (8)$$

which can also be written

$$-i(M_L{}^*-M_S)[g_L{}^\dagger g_S-(h_L{}^\dagger h_S)^*]=\Gamma_S g_L{}^\dagger g_S, \qquad (9)$$

a form very reminiscent of the Bell-Steinberger equation. Here Δ is the mass-difference angle,

$$\tan\Delta=\tfrac{1}{2}\Gamma_S/(m_L-m_S). \qquad (10)$$

Now consider symmetry constraints. If T alone were conserved, e.g., we could take $h=g$ for each resonance, so the bracketed factor of Eq. (9) would be pure imaginary, exactly like the $\langle L|S\rangle$ of Eq. (1).[2] Consequently, if we write

$$(g_L{}^\dagger g_S)=\rho e^{i\varphi}, \qquad (11)$$

we would have $\varphi\equiv\Delta$ (modπ).

If instead CPT is conserved, simple results can be obtained by choosing the channel states to be CP eigenstates, which is always possible. Then

$$\begin{aligned}T_{cc'}&=(\psi_c{}^{(+)},H\psi_{c'})=(CPT\psi_c{}^{(+)},HCPT\psi_{c'})^*\\&=(HCP\psi_{\hat{c}'},CP\psi_{\hat{c}}{}^{(-)})\\&=\pm(\psi_{\hat{c}'},H\psi_{\hat{c}}{}^{(-)})=\pm T_{\hat{c}'\hat{c}},\end{aligned}$$

where $\hat{c}$ and $\hat{c}'$ are the time-reversed states and the + (−) sign obtains if c and c' have the same (opposite) CP. Equivalently,

$$S_{cc'}=S_{\hat{c}'\hat{c}}. \qquad (12)$$

If magnetic quantum numbers are not included in the channel label c, we can equivalently write $S_{cc'}=\pm S_{c'c}$, in which case at a resonance Eq. (12) implies $g_{nc}h_{nc'}=g_{nc'}h_{nc}$, or

$$h_{nc'}/g_{nc'}=h_{nc}/g_{nc}\equiv\lambda_n^{+}, \tag{13}$$

for all c and c' which have, say, $CP=+1$. Since only the product $g_{nc}h_{nc'}$ appears in S, there is evidently no loss of generality in so defining them that $\lambda_n^{+}=\pm1$. Similarly λ_n^{-} can be defined, but if c and c' have opposite CP, Eq. (12) implies

$$h_{nc'}/g_{nc'}=-h_{nc}/g_{nc}, \tag{14}$$

i.e., $\lambda_n^{-}=-\lambda_n^{+}$.

Hence if CPT is valid for all the interactions, and in addition CP for the channel states, we can write in split notation

$$g_n=\begin{pmatrix} g_{n+} \\ g_{n-}\end{pmatrix},\quad h_n=\begin{pmatrix} \pm g_{n+} \\ \mp g_{n-}\end{pmatrix}, \tag{15}$$

where the "upper components" refer to $CP=+1$ channels.

If we call a resonance with the upper sign choice a resonance of "positive CP signature," then for two resonances which have opposite signature, we clearly have

$$h_1^{\dagger}h_2=-g_1^{\dagger}g_2.$$

Since K_S and K_L are nearly CP eigenvalues +1 and −1, it is most natural to adopt a phase convention which assigns positive signature to K_S and negative signature to K_L; this is the Bell-Steinberger convention, and the one which gives the Watson final-state theorem in its customary form. If this convention is employed, Eq. (8) or (9) determines the phase of $g_L^{\dagger}g_S$ to be $\varphi\equiv\Delta\pm\frac{1}{2}\pi$ (modπ). Since this is also the phase of the Bell-Steinberger $\sum$ when CPT is assumed we interpret this to mean that their sum rule and ours contain the same phase information, which is the only aspect employed phenomenologically.

Finally, we investigate the relation to the McGlinn-Polis sum rule only in the special case considered by them, in which the background matrix B is taken to be the unit matrix. $N-2$ of the N right eigenvectors of $S(E)$ are then orthogonal to h_L and h_S (in the non-Hermitian sense $\tilde{h}_Lv=\tilde{h}_Sv=0$), and all $N-2$ have S-eigenvalues unity; the remaining two eigenvectors lie in the subspace spanned by h_L and h_S. The matrix K (or T), being a function of S, <u>has these same eigenvectors</u>, with K-eigenvalues zero for the first $N-2$ eigenvectors. But since the McGlinn-Polis K is a sum of the dyadics $\Phi_L\Phi_L^{\dagger}$ and $\Phi_S\Phi_S^{\dagger}$, it can have the same eigenvectors as S only if $\tilde{h}_L$ and $\tilde{h}_S$ lie in the space spanned by $\Phi_L^{\dagger}$ and $\Phi_S^{\dagger}$. Similarly, g_L and g_S are linear combinations of Φ_L and Φ_S. It is then readily verified that the CPT condition, Eq. (15), can only be satisfied if

$$\Phi_L=\begin{pmatrix}\Phi_{L+}\\ i\Phi_{L-}\end{pmatrix},\quad \Phi_S=\begin{pmatrix}\Phi_{S+}\\ i\Phi_{L-}\end{pmatrix} \tag{16}$$

(aside from overall phase factors), with Φ_{L+}, Φ_{L-}, Φ_{S+}, and Φ_{S-} <u>real</u>, and

$$\begin{aligned} h_L&=\alpha_L\Phi_L^{*}+\alpha_S\Phi_S^{*},\\ g_L&=-\alpha_L\Phi_L-\alpha_S\Phi_S,\\ h_S&=\beta_L\Phi_L^{*}+\beta_S\Phi_S^{*},\\ g_S&=\beta_L\Phi_L=\beta_S\Phi_S. \end{aligned} \tag{17}$$

Note that the phase choice in Eq. (16) makes $\chi\equiv\Phi_S^{\dagger}\Phi_L$ <u>real.</u>

Finally, our S residues $-i\Gamma_L\,g_L\tilde{h}_L$ and $-i\Gamma_Sg_S\tilde{h}_S$ equal the McGlinn-Polis R_L and R_S (to first order in χ and Γ_L/Γ_S) only if

$$\begin{aligned} \alpha_L&=1,\quad \alpha_S=-i\Gamma_S\chi/2\delta m,\\ \beta_S&=i,\quad \beta_L=\Gamma_L\chi/2m\delta\approx0 \end{aligned} \tag{18}$$

(to within an overall minus sign), with $\delta m=m_L-m_S$. We note that this implies

$$g_L^{\dagger}g_S=-i\chi(1+i\,\Gamma_S/2\delta m), \tag{19}$$

which determines the phase of $g_L^{\dagger}g_S$ to be $\Delta\pm\frac{1}{2}\pi$ (modπ), in agreement with our above result.

Then, in the McGlinn-Polis notation,

$$\begin{aligned} \Gamma_L^{1/2}\langle n|H|\Phi_L\rangle&\equiv\langle n|\,2R_L|\Phi_L\rangle=2\Gamma_L\langle n|g_L\rangle\langle h_L|\Phi_L\rangle\\ &\approx2\Gamma_L\langle n|g_L\rangle,\\ \Gamma_S^{1/2}\langle n|H|\Phi_S\rangle&=\langle n|\,2R_S|\Phi_S\rangle\\ &=2\Gamma_S\langle n|g_S\rangle\langle h_S|\Phi_S\rangle\\ &\approx2i\Gamma_S\langle n|g_S\rangle. \end{aligned} \tag{20}$$

Consequently the complex conjugate of the McGlinn-Polis sum rule can be written in terms of these S-matrix amplitudes as

$$-i\chi\frac{\delta m}{\delta m-\frac{1}{2}i\Gamma_S}=4\sum_n\langle n|g_L\rangle^{*}\langle n|g_S\rangle, \tag{21}$$

from which it follows that the phase of this sum is also $\Delta\pm\frac{1}{2}\pi$ (modπ). We interpret this to mean that the K-matrix, S-matrix, and Bell-Steinberger sum rules are all equivalent in predicting the same phase for the unitarity sum of S-matrix decay amplitudes.

The remaining constraints in Eq. (7), combined with Eq. (15), impose many additional conditions on the decay amplitudes (especially for $K_L \to 2\pi$), which are discussed in the succeeding Letter.[7]

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S-MATRIX DESCRIPTION OF K_L AND K_S DECAYS*

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We show how the usual phenomenological description of the decays of the K_S and K_L mesons can be derived in a unified manner beginning from a description of the K_S and K_L states as overlapping resonances in a scattering matrix. The unitarity relations for overlapping resonances in a CPT-invariant (but not CP- or T-invariant) theory play a crucial role in the discussion, and are treated in detail.

In the present paper, we show how the usual phenomenological description of the decays of the neutral K mesons K_S and K_L can be derived in a simple, unified manner beginning from a description of the K_S and K_L states as overlapping resonances in a scattering matrix. The unitarity relations for the CPT-invariant S matrix are found to play a central role in the discussion of all decay modes of K_S and K_L. In the customary analysis,[1] on the other hand, unitarity is used only in the discussion of the CP-nonconserving decays, to determine the phase of the amplitude ratios

$$\epsilon = A(K_L \to \pi\pi,\ I=0)/A(K_S \to \pi\pi,\ I=0) \tag{1}$$

(through the Bell-Steinberger sum rule[1,2]), and

$$\epsilon' = A(K_L \to \pi\pi,\ I=2)/\sqrt{2}A(K_S \to \pi\pi,\ I=0) \tag{2}$$

(through the Watson final-state theorem applied to K and $\bar{K}$ decays). It does not enter the standard discussion of the semileptonic decay modes of K_L and K_S at all.

(a) General formulation.—The K_S and K_L mesons are overlapping resonances which decay into a common set of channels (predominantly, the 2π, 3π, $\pi l\bar{\nu}$, and $\pi\bar{l}\nu$ channels). If the energy dependence of the background scattering in these channels can be neglected in the neighborhood of the K_S and K_L masses, the partial-wave S matrix connecting the relevant channels can be approximated by the two-pole expression

$$S(E) = B - i\Gamma_S \frac{g_S \tilde{h}_S}{E-\xi_S} - i\Gamma_L \frac{g_L \tilde{h}_L}{E-\xi_L}. \tag{3}$$

The constant background matrix B describes that part of the scattering ($2\pi \to 2\pi$, $3\pi \to 3\pi$, etc.) not associated with K_S and K_L. ξ_S and ξ_L are the complex resonance energies for the K_S and K_L systems, $\xi_S = m_S - i\Gamma_S/2$ and $\xi_L = m_L - i\Gamma_L/2$. g_S and g_L are (constant) column vectors of partial-width amplitudes g_{Sc}, g_{Lc} for the decay of K_S and K_L into channel c, and $\tilde{h}_S$ and $\tilde{h}_L$ are the corresponding row vectors which describe the production of K_S and K_L through those channels. The usual decay and production amplitudes are related to the g's and h's by $A(K_S \to c) = \Gamma_S^{1/2} g_{Sc}$, $A(K_L \to c) = \Gamma_L^{1/2} g_{Lc}$, $A(c \to K_S) = \Gamma_S^{1/2} h_{Sc}$, $A(c \to K_L) = \Gamma_L^{1/2} h_{Lc}$.

The requirement that S be unitary throughout the K_S-K_L region leads to a unitarity relation for the background matrix B,

$$BB^\dagger = B^\dagger B = 1, \tag{4}$$