# TEST FOR TOLLER POLES

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The Treiman-Yang analysis is extended to test the validity of the O(3, 1) expansion of a two-body scattering amplitude and related to the $M$ value of a Toller pole.

We consider the high-energy limit of the five-particle production amplitudes of the type illustrated in Fig. 1 which is designed to illustrate a convenient choice of variables. Clearly,

$$k_1=p_0+p_1, \tag{1}$$

and

$$k_2=p_0+p_1+p_2=-(p_3+p_4). \tag{2}$$

The amplitude is a function of five independent scalar variables which can be taken to be

$$t_1=k_1^2, \quad t_2=k_2^2, \tag{3}$$

$$s=(p_1+p_2+p_3)^2=(p_0+p_4)^2, \tag{4}$$

$$s_1=(p_1+p_2)^2, \quad s_2=(p_2+p_3)^2. \tag{5}$$

By crossing, the amplitude may be given a variety of physical interpretations. In particular we consider the processes

$$p_1+p_2\to\bar{p}_0+\bar{p}_3+\bar{p}_4, \tag{a}$$

and

$$p_0+p_4\to\bar{p}_1+\bar{p}_2+\bar{p}_3, \tag{b}$$

where

$$\bar{p}_n=-p_n. \tag{6}$$

In process (a) $t_1$ is a momentum transfer, $t_2$ an energy squared, and $s_1$ the total energy ($>s$). In process (b) both $t_1$ and $t_2$ are momentum transfers and $s$ is the total energy ($>s_1, s_2$). We actually develop the amplitude for the process

$$p_0+p_1+p_2\to\bar{p}_3+\bar{p}_4, \tag{c}$$

in which both $t_1$ and $t_2$ are energy variables, and obtain the amplitudes for (a) and (b) by analytic continuation.

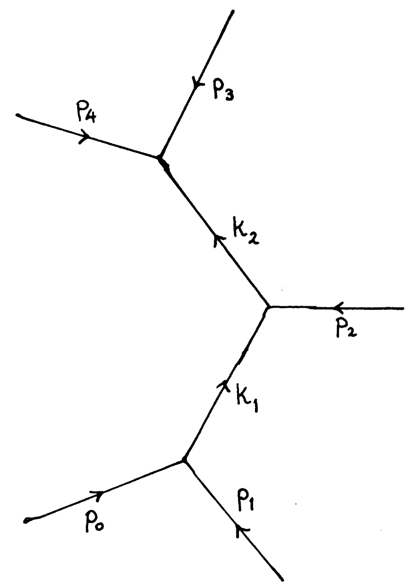


FIG. 1. The configuration of the five-particle amplitude which leads to the chosen sets of scalar variables.

In place of the variables $s_1$ and $s_2$ we can use $\theta_1$ and $\theta_2$ which are, respectively, the c.m.-system scattering angles in the virtual subprocesses

$$p_0+p_1\to k_2+\bar{p}_2 \tag{7}$$

and

$$k_1+p_2\to\bar{p}_3+\bar{p}_4. \tag{8}$$

In place of $s$ we can use $\varphi$ which is the angle between the above two scattering planes. The amplitude for process (c) can be written[1,2] in terms of explicit O(3) functions of $\theta_1$, $\theta_2$, and $\varphi$ as

$$T=\sum_{J_1,J_2,\Lambda_1} g_3(J_2)d_{\lambda_3-\lambda_4,\Lambda_2}{}^{J_2}(\theta_2)\exp(i\Lambda_1\varphi)g_2(J_2,J_1,\Lambda_1)d_{\Lambda_1\lambda}{}^{J_1}(\theta_1)g_1(J_1), \tag{9}$$

where

$$\Lambda_2=\Lambda_1+\lambda_2, \quad \lambda=\lambda_1-\lambda_0, \tag{10}$$

$j_1$ and $\lambda_i$ are, respectively, the spin and spin components of the $n$th particle,[3] and the summation is over the ranges $J_1\geq|\lambda_1-\lambda_0|$, $J_2\geq|\lambda_3-\lambda_4|$, and

$$|\Lambda_1|\leq J_1, \tag{11}$$

$$|\Lambda_2|\leq J_2. \tag{12}$$

In Eq. (19) the $g_i$ are vertex factors which depend on the variables $t$ and the masses, spins, and spin components of the particles which couple at the vertex. The entire dependence of the amplitude on $s_1$, $s_2$, and $s$ is through the angles $\theta_1$, $\theta_2$, and $\varphi$.

The unit vectors perpendicular to the planes defined by the virtual subprocesses (7) and (8) are

$$(\beta_1)_\mu = \epsilon_{\mu\nu\lambda\rho} p_0^{\ \nu} p_1^{\ \lambda} p_2^{\ \rho}/\Phi^{1/2}(s_1 t_1; m_0^2, m_1^2, m_2^2, t_2) \tag{13}$$

and

$$(\beta_2)_\mu = \epsilon_{\mu\nu\lambda\rho} p_2^{\ \nu} p_3^{\ \lambda} p_4^{\ \rho}/\Phi^{1/2}(s_2, t_2; t_1, m_2^2, m_3^2, m_4^2), \tag{14}$$

where $\Phi$ is the Kibble function[4] for the appropriate process ($\Phi = 0$ is the boundary for physical regions). Then

$$\cos\varphi = -\beta_1 \cdot \beta_2. \tag{15}$$

Since the four-vector $p_n$ is always timelike the vectors $\beta_1$ and $\beta_2$ are spacelike for any physical region of the amplitude, and consequently $\varphi$ is always a real angle in any physical region.

For process (a), Treiman and Yang[5] made the important observation that if the sum over $J_1$ in (9) is dominated by a single resonance (Feynman pole) of spin $S_1$, then by (11) the summation over $\Lambda_1$ is restricted to

$$|\Lambda_1| \leq S_1. \tag{16}$$

This in turn severely restricts the dependence of $T$ on $\varphi$ which in this context is known as the Treiman-Yang angle. In particular, if $S_1 = 0$, this shows that the amplitude is independent of $\varphi$. This restriction was proposed as a test of the peripheral model for the production of a resonance of mass $t_2$ [in the $s_1$ channel of subprocess (7); $p_1 + p_2 \to \bar{p}_0 + k_2$], which subsequently decays into particles $\bar{3}$ and $\bar{4}$.[6]

We now generalize this argument to show how observations of the $\varphi$ dependence can be used to test O(3, 1) expansions of scattering amplitudes and to determine the quantum numbers of Toller poles. We remark that the last three factors of (9) (summed over $J_1$) constitute that $t_1$-channel O(3) expansion of the scattering amplitude[7] for the subprocess (7):

$$\sum\nolimits_{J_1} g_2(J_2, J_1, \Lambda_1) d_{\Lambda_1,\lambda}^{J_1}(\theta_1) g_1(J_1) \equiv \langle J_2, \Lambda_2, j_2, \lambda_2 | T_1(s_1, t_1) | j_0, \lambda_0, j_1, \lambda_1 \rangle \equiv T_1. \tag{17}$$

This amplitude can alternatively be expanded in terms of O(3, 1) functions[8,9];

$$T_1 = \sum d_{\Lambda_1,m}^{J}(\theta^L) d_{J,m,j}^{M_1\sigma_1}(\epsilon) d_{m,\lambda}^{j}(\theta^R) T_{J,j}^{M_1,\sigma_1}(t_1; J_2, \Lambda_2, j_2, \lambda_2, j_1, \lambda_1, j_0, \lambda_0), \tag{18}$$

where $(M_1, \sigma_1)$ specify an O(3, 1) representation. The summation is over the ranges

$$|M_1|, |m| \leq \min(J, j), \tag{19}$$

and

$$|j_0 - j_1| \leq j \leq j_0 + j_1, \tag{20}$$

$$|J_2 - j_2| \leq J \leq J_2 + j_2, \tag{21}$$

and it is, of course, assumed that there is some range of $\sigma$ for which the O(3, 1) expansion represents the amplitude in the region in which $s_1$ is an energy variable and $t_1$ a momentum transfer [i.e., processes (a) and (b)]. Further,[10]

$$\cosh\epsilon = p_1 \cdot p_2 / m_1 m_2, \tag{22}$$

$$\theta^L = \theta(k_1, p_1; p_2), \quad \theta^R = \theta(k_1, p_2; p_1), \tag{23}$$

and conservation of parity requires that the summation be over the representations $|M|, \sigma_1$ and $-|M_1|, \sigma_1$.

We now consider (18) in the limit $s_1 \to \infty$ and $t_1$ small which is in the physical region of processes (a) and (b). In this limit the behavior of the $d$ functions in (18) is[11]

$$s_1^{\sigma_1 - 1|m - M_1|} t^{|\Lambda_1 \mp m|/2} t^{|\lambda \mp m|/2}. \tag{24}$$

If the range of real values of $\sigma_1$ is bounded and if the limit $s_1 \to \infty$ can be taken inside the summation in

(18), the leading $s_1$ dependence of $T_1$ is given by that term in the summation for which[12]

$$m = M_1. \tag{25}$$

Substituting this behavior of $T_1$ into (9) we see that the leading behavior for small $t_1$ comes from that term in the summation over $\Lambda_1$ for which

$$\Lambda_1 = \pm|M_1|. \tag{26}$$

This is turn implies that the leading behavior of $T$ in this limit is

$$T_{s_1\to\infty, t_1\to 0} = \sum_{M_1=O(1/2)}^{j_0+j_1} s_1^{\sigma_1 - 1}\{f_M(s_2,t_2)\exp(iM_1\varphi) + f_{-M}(s_2,t_2)\exp(-iM_1\varphi)\}, \tag{27}$$

which is the generalization of the Treiman-Yang result.[13] Clearly if the expansion of $T_1$ is dominated by a single Toller pole with quantum number $M_1$ this completely determines the $\varphi$ dependence of the amplitude in the high-$s_1$ limit ($M_1$ plays a role very similar to the spin $S_1$ of the Feynman pole in the Treiman-Yang argument). In particular, if the $M$ value of the Toller pole is zero the amplitude in this limit is independent of $\varphi$.

For resonance production processes of type (a) the information required from the experimenter is exactly the same as for the Treiman-Yang analysis. Only the interpretation is different. For example, in the process

$$\pi N \to N\rho \to N\pi\pi$$

independence of the Treiman-Yang angle now indicates the exchange of a leading Toller pole with $M = 0$, rather than the peripheral exchange of a pion; an observed dependence in (27) of $M_1 > 1$ would cast doubt on the validity of O(3, 1) expansions.

Similar considerations apply to process (b), where now, however, $t_2$ is spacelike and the relevant angle $\varphi$ is the angle between the $(p_0, \bar{p}_1)$ and the $(p_4, \bar{p}_3)$ planes in the frame in which $\vec{\bar{p}}_2 = 0$.

Present experimental data are not inconsistent with such O(3, 1) expansions. See, e.g., Berger.[14]

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[1]N. F. Bali, G. F. Chew, and A. Pignotti, Phys. Rev. 163, 1572 (1967); M. Toller, Nuovo Cimento 37, 731 (1965).

[2]G. Feldman and P. T. Matthews, "The High Energy Limit of Production Amplitudes" (to be published), and for details of such expansions, their work entitled "General Expansions of a Scattering Amplitude" (to be published).

[3]The spin components $\lambda_0$ and $\lambda_1$ are the helicities of particles "0" and "1," respectively, in the $(p_0, p_1)$ c.m. frame (i.e., $\vec{k}_1 = 0$) and $\lambda_2, \lambda_3, \lambda_4$ are the helicities of particles "2", "$\bar{3}$," and "$\bar{4}$" in the $(\bar{p}_3, \bar{p}_4)$ c.m. frame (i.e., $\vec{k}_2 = 0$). See Feldman and Matthews, Ref. 2, for details, where however in place of process (c) the formation process $p_0 + p_1 + p_2 + p_3 \to \bar{p}_4$ is discussed.

[4]T. W. B. Kibble, Phys. Rev. 117, 1159 (1960).

[5]S. B. Treiman and C. N. Yang, Phys. Rev. Letters 8, 140 (1962).

[6]Two generalizations of this argument have been given which are essentially trivial if we recognize alternative interpretations of the angles $\varphi$ and $\theta_2$. From the covariant definition of $\varphi$ given in (15) and the conditions (1) and (2) it is immediately obvious that $(\theta_2, \varphi)$ are the polar angles of the vector $\vec{\bar{p}}_3$ in the rest frame of the "decaying" particle $k_2$, where the $z$ axis is the direction of $\vec{\bar{p}}_2$ and the $y$ axis is the direction of $\vec{p}_1 \times \vec{\bar{p}}_0$ in this frame. See J. D. Jackson, Nuovo Cimento 34, 1644 (1964). Thus if in process (a) the summation over $J_2$ is dominated by a single resonance of spin $S_2$ (the spin of the produced particle in the Treiman-Yang argument), then by (12) we have the additional restriction $|\Lambda_2| \leq S_2$. See K. Gottfried and J. D. Jackson, Nuovo Cimento 33, 310 (1964). The second generalization is to the process (b) in the multi-Regge model in which the summations over $J_1$ and $J_2$ in (9) are analytically continued by a Sommerfeld-Watson transform and assumed to be dominated by Regge poles at $\alpha_1(t_1)$ and $\alpha_2(t_2)$. [See T. W. B. Kibble, Phys. Rev. 131, 2282 (1963).] For values of $t_1$ which correspond to a physical particle for which $\alpha_1(t_1) = S_1$, by (11) the dependence on $\varphi$ in (9) must again satisfy the condition (16), $|\Lambda_1| \leq S_1$. I. T. Drummond, in Proceedings of the International Conference on High Energy Physics, Lund, Sweden, 1969 (to be published).

[7]The amplitude (17) is not precisely the $t_1$-channel helicity amplitude for subprocess (7). Although $\lambda_0$ and $\lambda_1$ are the correct helicity labels, $\Lambda_2$ and $\lambda_2$ are spin components in the rest frame of "particle" $k_2$. However, the choice of axis is such that a pure boost in the direction of $\vec{\bar{p}}_2$ in this frame takes one to the "c.m." frame $\vec{k}_1 = 0$, and thus $\Lambda_2$ and $\lambda_2$ are the correct c.m.-frame helicity labels. Note that the first three factors in (9) represent the decay of the "$k_2$ resonance". The amplitude for a process of type (a) was written in this form as a product of a scattering

amplitude (17) and a decay amplitude with these helicity labels by Gottfried and Jackson, Ref. 6.

[8]R. Delbourgo, A. Salam, and J. Strathdee, Phys. Rev. 164, 1981 (1967); G. Cosenza, A. Sciarrino, and M. Toller, Nuovo Cimento 57, 253 (1968); G. Domokos and G. L. Tindle, Phys. Rev. 165, 1906 (1968); W. R. Frazer, F. R. Halpern, H. M. Lipinski, and D. R. Snider, Phys. Rev. 176, 2047 (1968).

[9]See also Feldman and Matthews, Ref. 2, part 2, for a general review and further references.

[10]If $a$, $b$, and $c$ are any four-vectors ($c$ timelike), we denote by $\theta(a,b;c)$ the angle between $\vec{a}$ and $\vec{b}$ in the frame in which $\vec{c}=0$. (See, e.g., Feldman and Matthews, Ref. 2, part 2, Appendix.)

[11]See, e.g., K. M. Bitar and G. L. Tindle, Phys. Rev. 175, 1835 (1968) or Feldman and Matthews Ref. 2, part 2.

[12]We clearly also have the restriction $\lambda = M_1$, which determines which of the helicity amplitudes of the $p_1, p_0$ system survives at $t_1 = 0$.

[13]If the sum over $J_2$ in (9) is dominated by a single resonance of spin $S_2$, then $M_1 \leq \min(j_0+j_1, S_2+j_2)$.

[14]E. L. Berger, Phys. Rev. Letters 21, 701 (1968).

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# VENEZIANO-TYPE FORM FACTORS FOR THE PION*

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The scalar, pseudoscalar, vector, and axial-vector form factors of the pion are derived from (i) the Veneziano amplitudes for $\pi^+\pi^- \to \pi^+\pi^-$ and $\pi^+\pi^- \to A_1{}^+\pi^-$, (ii) the hypothesis of the partially conserved axial-vector current, and (iii) the requirement that a chirally conjugate pair of form factors should exhibit similar structures. The resulting form factors feature an infinite sequence of poles corresponding to daughters of specific spin and parity, and definite signature. Various predictions on the clashing-beam production of $2\pi$ and on $\rho$-meson coupling constants are given.

Recently, the Veneziano-type form factors for the pion have been discussed by several authors,[1] but these fail to satisfy a natural requirement that a chirally conjugate pair of sources, like the vector current $V_\mu{}^a$ and the axial-vector current $A_\mu{}^a$, should couple in more or less symmetric ways to an infinite number of particles with specific spin and parity, whose existence is one consequence of the Veneziano model.[2]

To illustrate our point, let us take for example the following off-shell $\pi\pi$ scattering amplitude which is consistent with the condition of partially conserved axial-vector current (PCAC) and which reduces to the Veneziano-Lovelace[3] amplitude on the mass shell:

$$\langle \pi^-(q')|\partial^\lambda A_\lambda{}^{(-)}(0)|\pi^+(p), \pi^-(q)\rangle = \sqrt{2} f_\pi m_\pi{}^2 (k^2 - m_\pi{}^2)^{-1} \beta_0 B_1{}^{1,1}(s,t), \tag{1}$$

where

$$B_k{}^{i,j}(s,t) \equiv \Gamma(i-\alpha_s)\Gamma(j-\alpha_t)/\Gamma(k-\alpha_s-\alpha_t).$$

Here $k$ is the momentum of the off-shell $\pi^+$ meson and $s=(p+q)^2$, $t=(k-p)^2$, and $u=(q-k)^2$. $\alpha_s = \alpha(s)$ represents the degenerate $\rho$-$f$ trajectory and is given by $\alpha(s) = \frac{1}{2} + b(s - m_\pi{}^2)$ with $b^{-1} = 2(m_\rho{}^2 - m_\pi{}^2)$. $f_\pi$ is the pion decay constant and $\beta_0 = g_{\rho\pi\pi}{}^2$. Now by continuing to $p \to 0$ by means of the standard soft-pion technique the left-hand side of (1) reduces to the pion matrix element of the so-called $\Sigma$ term,

$$[A_0{}^{(+)}(x), \partial^\lambda A_\lambda{}^{(-)}(0)]_{x_0=0} = 2i\delta^3(x)\Sigma(0).$$

On the right-hand side, we have $s=u=m_\pi{}^2$, $t=k^2$, and using $\alpha(m_\pi{}^2) = \frac{1}{2}$ we obtain[4]

$$\langle \pi^-(q')|\Sigma(0)|\pi^-(q)\rangle \equiv \tilde{\Sigma}(t) = -\frac{m_\pi{}^2}{\sqrt{\pi}} \frac{\Gamma(1-\alpha_t)}{\Gamma(\frac{3}{2}-\alpha_t)}. \tag{2}$$

Here we have used a scaling law[5]

$$\pi f_\pi{}^2 \beta_0 b = 1 \tag{3}$$

in order to obtain the above normalization. Thus, in spite of our original assumption expressed in (1), namely, that the pseudoscalar source $\partial^\lambda A_\lambda{}^{(-)}(x)$ couples only to the $\pi$ meson, we obtain the result that its chiral counterpart $\Sigma$ couples to all $0^+$ daughters of the $\rho$-$f$ trajectory. We consider this asym-