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HARMONIC-OSCILLATOR ANALOGY FOR THE VENEZIANO MODEL*

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A model for particle scattering amplitudes is based on the harmonic-oscillator Green's function. The model is Regge behaved, and in first approximation is a zero-width theory. The derived amplitudes are very similar to Veneziano n-point functions although they lack duality.

We present a model scattering matrix based on a relativistic harmonic oscillator. The interest in the model stems from its similarity to the Veneziano model in the following respects: (i) It contains an infinite spin-mass spectrum identical to the Veneziano model. However, it should be remarked that the degeneracy at each daughter site is probably different. (ii) The lowest order of perturbation theory is a zero-width approximation. (iii) The model is multi-Regge behaved. (iv) By appropriate choice of a single parameter the coupling scheme of the leading trajectory is identical to that in the Veneziano model. (v) The Chan[1] representation for the n-point function is modified in a remarkably simple manner in the oscillator model.

Questions of renormalization, finite-width corrections, off-shell continuations, and local currents in the model are under investigation by Frye, Gallardo, and the author.

Consider the Bethe-Salpeter equation for a quark-antiquark pair,

$$(\Box_1 - m^2)(\Box_2 - m^2)\psi(x_1, x_2) = U(x_1, x_2)\psi. \qquad (1)$$

Letting $m^2 \to \infty$ so that $U/m^2 - m^2$ remains finite and making a change of variables to $X = \frac{1}{2}(x_1 + x_2)$ and $x = (x_1 - x_2)$ gives

$$[\tfrac{1}{2}\Box_X + 2\Box_x + V(x)]\psi(x, X) = 0. \qquad (2)$$

A solution with total four-momentum p has the form $e^{ipX}\varphi(x)$. Inserting this in Eq. (2) and performing a Wick rotation gives the O(4)-symmetric equation

$$\left(-\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\mu}+\tfrac{1}{2}V\right)\varphi=\tfrac{1}{4}p^2\varphi. \tag{3}$$

We choose $\frac{1}{2}V(x)=x^2$ to give the four-dimensional oscillator equation and the mass quantization condition $\frac{1}{4}p^2=\frac{1}{4}M^2=2n+1$. The solutions are parametrized by four excitation integers n_1, n_2, n_3, and n_4 and are of the form

$$\varphi_n(x)=e^{-x^2/2}\prod_{i=1}^{4}H_{n_i}(x_i)(2^{n_i}n_i!)^{-1/2}, \tag{4}$$

or equivalently, in momentum space[2]

$$\varphi_n(\bar{p})=\prod e^{-\bar{p}_i{}^2/2}H_n(\bar{p}_i)(2^{n_i}n_i!)^{-1/2}, \tag{5a}$$

where $\bar{p}=\frac{1}{2}(p_1-p_2)$ and $\bar{p}_i$ is its ith component. Using the generating function for H_n gives

$$\varphi_n(\bar{p})=e^{-\bar{p}^2/2}\prod\frac{(\partial/\partial\alpha_i)^{n_i}e^{-\alpha_j\alpha_j+2\alpha_j\bar{p}_j}}{(n_i!2^n)^{1/2}}\Bigg|_{\alpha=0}. \tag{5b}$$

Assume now that the quarks are coupled to a scalar neutral massless field. The vertex connecting two states of the oscillator and an emitted field quantum is constructed from the graph in Fig. 1 with the following rules: The bubble vertices are replaced by a wave function φ_n which is best expressed in the form of Eq. (5b), a point vertex is a coupling constant, and a quark line usually given by $1/(k^2-m^2)$ is a constant in the limit $m^2\to\infty$. The vertex is then given by a generating function of two four-vectors, $F(\alpha,\beta)$, such that the transition between the states n_i and m_j is

$$\prod\left(\frac{\partial}{\partial\alpha_i}\right)^{n_i}\left(\frac{\partial}{\partial\beta_j}\right)^{m_j}F(\alpha,\beta)(2^{n_i+m_j}n_i!m_j!)^{-1/2}. \tag{6}$$

The integrations implied by the graph in Fig. 1 are all Gaussian integrals and yield

$$F(\alpha,\beta)=\exp[(\alpha+\beta)k/2-2\alpha\beta] \tag{7}$$

for k satisfying the mass-shell condition $k^2=0$.

Next consider the scattering of a field quantum as shown in Fig. 2. The rules for such a graph are to insert the expression (7) at the oscillator-quantum vertices and sum over intermediate oscillator states with the factor $1/(p^2-8n-4)$. For external oscillators in the ground state this gives

$$T=\sum_n e^{\alpha k_1/2}\left(\frac{\vec{\partial}}{\partial\alpha}\cdot\frac{\vec{\partial}}{\partial\beta}\right)^n e^{\beta\cdot k_2/2}\frac{(2^n n!)^{-1}}{p^2-8n-4}=\sum_n\left(\frac{k_1\cdot k_2}{8}\right)^n\frac{1}{(p^2-8n-4)n!}. \tag{8}$$

It is convenient to eliminate the 8 by change of momentum scale. Equation (8) can be summed by use of the identity

$$\int_0^1 X^A dX=(A+1)^{-1}$$

to give

$$\int_0^1 X^{-p^2-1/2}(e^{-x})^{-t_{12}/2}dX, \tag{9}$$

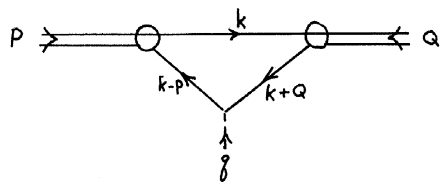

FIG. 1. Kinematics for the oscillator-quantum vertex.

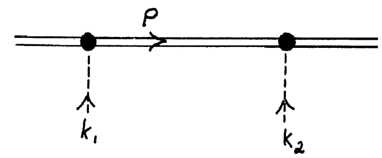

FIG. 2. Kinematics for the four-point function.

where $t_{12}=2k_1\cdot k_2$, which is very similar to the Veneziano amplitude in which the $(e^{-x})^{-t_{12}/2}$ is replaced by $(1-x)^{-t_{12}}$.

Actually the $t_{12}/2$ can be replaced by t_{12} as the ratio of the two quark masses is not 1.

The five-point function is similarly evaluated. The kinematics is shown in Fig. 3. Again a change of momentum scale has been made.

$$T=\sum_{m,n}e^{-\alpha k_1}\left(\frac{\overleftarrow{\partial}}{\partial\alpha}\cdot\frac{\overrightarrow{\partial}}{\partial\beta}\right)^n e^{(\beta+\gamma)k_2-2\beta\gamma}\left(\frac{\partial}{\partial\gamma}\cdot\frac{\partial}{\partial\delta}\right)^m e^{\delta\cdot k_3}\frac{1}{n!m!2^{n+m}(p_1^2-n-\frac{1}{2})(p_2^2-m-\frac{1}{2})}$$

$$=\int X_1^{-p_1^2-1/2}X_2^{-p_2^2-1/2}\left(\frac{e^{-x_1}}{e^{-x_1x_2}}\right)^{-k_1\cdot k_2/2}\left(\frac{e^{-x_2}}{e^{-x_1x_2}}\right)^{-k_2\cdot k_3/2}(e^{-x_1x_2})^{-(k_1+k_2+k_3)^2/2}. \tag{10}$$

This is almost the Veneziano five-point function. The replacement of the $e^{-x\cdots}$ by $(1-x\cdots)$ would make them identical.

The following generalization can be proved: If the Chan representation[1] of the n-point function is written as

$$\int X_1^{-s_1}\cdots X_n^{-s_n}T_l^{-t_1}\cdots T_m^{-t_m}d^nX$$

with $(X_1\cdots X_n)$ being a complete set of independent dual parameters, and if the T's are given by

$$T_a=\frac{(1-X_iX_j\cdots)(1-X_kX_l\cdots)}{(1-X_mX_n\cdots)(1-X_r\cdots)}, \tag{11}$$

then the corresponding oscillator scattering amplitude is related by changing Eq. (11) to

$$T_a=\frac{(e^{-X_iX_j\cdots})(e^{-X_kX_l\cdots})}{(e^{-X_mX_n\cdots})(e^{-X_r\cdots})}. \tag{12}$$

Actually this model is quite poor. There are an infinite number of ghost poles probably due to the action-at-a-distance nature of the oscillator force. We conjecture that this disease can only be removed by eliminating the action at a distance by allowing the force to be transmitted through some sort of continuum. For example we might replace the single "spring" by the continuum limit of a chain of springs. The effect will be to increase drastically the degeneracy of the levels. For example, for a chain of springs fixed at one end with Hamiltonian $\sum\dot{x}_i^2+(x_i-x_{i-1})^2$, the problem is still separable in the four directions of oscillation. However, instead of a single creation operator and occupation number for each direction we have a countable infinity corresponding to the fundamental mode and all its harmonics. If n_i is the occupation number for the ith harmonic, the energy for a given direction is $\sum in_i$. If we consider a given level n, there are many more ways to excite energy n. The simplest is the set of states formed by the action of n creators for the fundamental modes. The set of such states forms the space of symmetric tensors of rank n and therefore carries angular momentum $\leqslant n$. The other ways of exciting energy n involve fewer creation operators and therefore correspond to tensors of rank less than n. Hence the huge degeneracy implied by such a model effects only daughter poles. Since it is not known how degenerate the daughters in the Veneziano model have to be we cannot rule out such a model. It is of great interest to determine what degeneracy is required to cause the daughter poles to factorize.

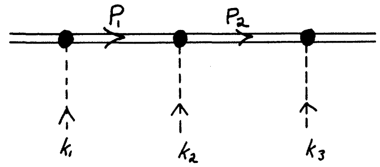

FIG. 3. Kinematics for the five-point function.

The author acknowledges the help of Professor Graham Frye in some of the calculations reported here.

Note added in proof.—The problem of factorization of Veneziano amplitudes has recently been solved by Fubini and Veneziano,[3] Bardakçi and Mandelstam,[4] and this author.[5] The solutions agree exactly with the form of spectrum postulated on the basis of a harmonic continuum model with cyclic boundary conditions, or in other words, a rubber band.

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