RELATIVISTIC EIKONAL EXPANSION

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(Received 19 May 1969)

A fully relativistic eikonal expansion is discussed. The connection with the high-energy behavior of elastic scattering, especially for quantum electrodynamics, is examined, and the relevance of the results for Regge asymptotic behavior is investigated.

In an interesting paper in this journal Cheng and Wu[1] have considered the high-energy behavior of scattering processes in quantum field theory. They examined certain sets of Feynman graphs for spinor quantum electrodynamics which may be described as generalized ladders (they include all manner of crossed graphs also) in the channel whose energy is being taken large. Subsequently, Chang and Ma[2] have demonstrated, by use of very clever techniques for exhibiting the high-energy behavior of perturbation-theory graphs, that the subsets of graphs under consideration can be summed to the eikonal form of Moliere and Glauber.[3] This is consistent with the previous examination of the same set of graphs by Torgerson.[4]

We would like to show here a straightforward method for deriving the results of Refs. 1 and 2 which clearly exhibits the "eikonal" nature of the high-energy approximation and indicates how one may systematically correct this approximation. The technique we employ is that of functional derivatives which has been developed by Schwinger.[5] This procedure allows one to emphasize the striking similarity between the eikonal expansions in the nonrelativistic and relativistic cases.

Let us consider, then, spinless particles of mass m (we will discuss the transition to spinor electrodynamics later) undergoing an elastic collision, with forces mediated by spinless objects of mass μ. The sum of all Feynman graphs of the "ladder" type as shown in Fig. 1 leads to the $\mathcal{T}$-matrix element

$$\frac{-i}{(2\pi)^4}\mathcal{T}(p_2',p_2;p_1',p_1)\delta^4(p_2+p_2'-p_1-p_1')=\lim_{p_i^2\to m^2}(p_1^2-m^2)(p_2^2-m^2)(p_3^2-m^2)(p_4^2-m^2)\mathcal{K}\langle p_2|G(A)|p_1\rangle$$
$$\times\langle p_2'|G(A')|p_1'\rangle|_{A=A'=0}, \qquad (1)$$

where $G(A)$ is the Green's function for a spinless particle in an external scalar potential (source) A:

$$G^{-1}(A)=P^2-m^2-A(X)+i\epsilon, \qquad (2)$$

while the P and X are four-dimensional noncommuting operators satisfying $[X_\mu, P_\nu]=ig_{\mu\nu}$. Finally $\mathcal{K}$ expresses the exchange of the intermediate boson,

$$\mathcal{K}=\exp\iint d^4y\,d^4y'\frac{\delta}{\delta A(y)}D(y-y')\frac{\delta}{\delta A'(y')}, \qquad (3)$$

with $D(x)$ the usual causal propagator

$$D(x)=\frac{ig^2}{(2\pi)^4}\int\frac{d^4k\,e^{-ik\cdot x}}{k^2-\mu^2+i\epsilon}. \qquad (4)$$

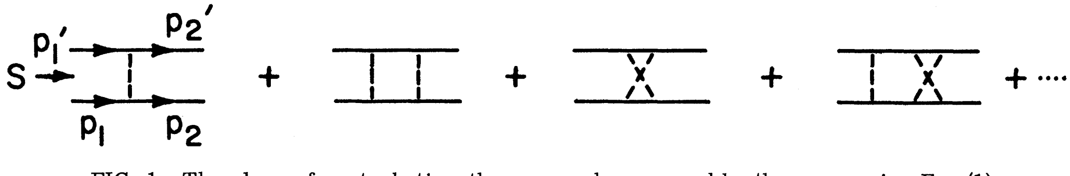

FIG. 1. The class of perturbation-theory graphs summed by the expression Eq. (1).

This form for the $\mathcal{T}$ matrix is only a convenient bookkeeping for the sum of Feynman graphs it represents. It is cast, however, in a manner conducive to the displaying of the eikonal expansion.

Using the operator identity[6] $\exp(A+B)=(\exp A)T(\exp\int_0^1 dt\, e^{-At}Be^{At})$, we can write

$$\frac{-i}{(2\pi)^4}\mathcal{T}(p_2',p_2;p_1',p_1)\delta^4(p_2'+p_2-p_1'-p_1)=\mathcal{K}\langle p_2|\mathcal{T}(A)|p_1\rangle\langle p_2'|\mathcal{T}(A')|p_1'\rangle|_{A=A'=0},\tag{5}$$

with

$$\mathcal{T}(A)=T\{\exp[-i\int_0^\infty A(X-2P\tau)d\tau]\}A(X),\tag{6}$$

and a similar term for $\mathcal{T}(A')$. The guiding principle of the eikonal approximation is that a very energetic particle can hardly be deflected from its path by reasonable interactions. It is natural then to attempt a perturbation expansion around the direction of the high-energy particle where the perturbation terms will be, essentially, responsible for its deviation from a straight path.

The first term in the eikonal expansion is obtained by replacing the operators P and P' by $p=\frac{1}{2}(p_1+p_2)$ and $p'=\frac{1}{2}(p_1'+p_2')$. The corresponding eikonal $\mathcal{T}$ matrix $\mathcal{T}_E$ is given by

$$\begin{aligned}\langle p_2|\mathcal{T}_E(A)|p_1\rangle&=\int\frac{d^4x}{(2\pi)^4}e^{i(p_1-p_2)\cdot x}\exp[-i\int_0^\infty A(x-2p\tau)d\tau]A(x)\\&=\int\frac{d^4x}{(2\pi)^4}e^{i(p_1-p_2)\cdot x}\left(\frac{d}{id\alpha}\exp[-i\int_\alpha^\infty A(x-2p\tau)d\tau]\right)_{\alpha=0}.\end{aligned}\tag{7}$$

Carrying out the functional derivatives implied by $\mathcal{K}$, by noting it is a shift operator in A and A', leads at once to

$$\begin{aligned}&\frac{-i}{(2\pi)^4}\mathcal{T}_E(p_2',p_2;p_1',p_1)\delta^4(p_2+p_2'-p_1-p_1')\\&=-\int\frac{d^4x d^4x'}{(2\pi)^8}e^{i(p_1-p_2)\cdot x+i(p_1'-p_2')\cdot x'}\left(\frac{d}{d\alpha}\frac{d}{d\alpha'}\exp-i\int_\alpha^\infty d\tau\int_{\alpha'}^\infty d\tau' D(x-x'-2p\tau+2p'\tau')\right)_{\alpha=\alpha'=0}.\end{aligned}\tag{8}$$

This is analogous to the case of potential scattering with $D(x)$ as a generalized "potential." Note that relativistic covariance has been maintained throughout.

Now energy-momentum conservation can be factored out by integration over $x+x'$. Next notice that $p_1-p_2=-(p_1'-p_2')$ is orthogonal to both p and p' so we may set $x-x'=b-2p\sigma+2p'\sigma'$, where b is a two-dimensional vector. Two integrations may then be carried out to cast (8) into its final form:

$$\mathcal{T}_E(s,t)=-2i\bar{s}\int d^2b\, e^{-i(\vec{p}_1-\vec{p}_2)\cdot\vec{b}}\left\{\exp\left[\frac{ig^2}{2\bar{s}}\int\frac{d^2q}{(2\pi)^2}\frac{e^{-i\vec{q}\cdot\vec{b}}}{|\vec{q}|^2+\mu^2}\right]-1\right\},\tag{9}$$

where $(\vec{p}_1-\vec{p}_2)^2=-t$, and $\bar{s}=s[1-(t+4m^2)/s]^{1/2}$, which is effectively s.

A few remarks are now in order about this leading eikonal expression. (1) To appreciate (9) it is good to remember that it can be thought of as a resummation of the leading terms in s to each order in $(g^2)^n$ of the sum of corresponding Feynman graphs. Cancellation among diagrams of the same order occurs. (2) In the scalar case we have been investigating it can easily be shown that in (9) the first Born approximation (behaving as a constant as $s\to\infty$) indeed is the leading term. This is in precise analogy to potential scattering. (3) In the present approach the addition of self-interactions or "radiative corrections" amounts to adding diagonal terms to the functional differentiation operator $\mathcal{K}$. One might imagine, after performing the usual mass and charge renormalizations, evaluating such effects. (4) The "potential" $D(x)$ can clearly be modified to include the exchange of any spin particle or to exhibit the effects of "vacuum polarizations" on the exchanged line—all without any alteration in the preceding argument. (5) In the interesting case of vector exchange the eikonal expansion becomes indeed relevant since all terms in the set of graphs become comparable. In electrodynamics of spinor particles one replaces (2) by

$$\frac{\not{P}+m-e\not{A}(X)}{[P-eA(X)]^2-m^2+i\epsilon}$$

and obtains a result only slightly modified from (9) as the following heuristic argument shows. At high energy only the convective part of the current survives, so that in the case of charged spin-$\frac{1}{2}$ particles no spin flip occurs. This amounts to saying that exchanging a photon of virtual momentum q yields a factor $-(ee')4p'\cdot p[q^2-\mu^2+i\epsilon]^{-1}$ (μ is an infrared cutoff), instead of $g^2[q^2-\mu^2+i\epsilon]^{-1}$ in the scalar case. The asymptotic form of the series of perturbation graphs of Fig. 1 for spinor electrodynamics is then

$$\mathcal{T}_E(s,t) = -2is\frac{\delta_{\lambda_1\lambda_2}\delta_{\lambda_1'\lambda_2'}}{4m^2}\int d^2b\, e^{i\vec{b}\cdot(\vec{p}_2-\vec{p}_1)}\left\{\exp\left[-iee'\int\frac{d^2q}{(2\pi)^2}\frac{e^{-i\vec{q}\cdot\vec{b}}}{|\vec{q}|^2+\mu^2}\right]-1\right\}, \tag{10}$$

where the λ_i are the particle helicities, and a conventional normalization of Dirac particles has been inserted. For $\mu\to 0$ this result can be expressed, apart from the usual phase, as

$$\mathcal{T}_E(s,t) = -\frac{\pi s}{m^2}\delta_{\lambda_1\lambda_2}\delta_{\lambda_1'\lambda_2'}\left(\frac{ee'}{4\pi}\right)\frac{\Gamma(1+(iee'/4\pi))}{\Gamma(1-(iee'/4\pi))}\left(\frac{-t}{2}\right)^{-[1+(iee'/4\pi)]}, \tag{11}$$

which displays—as is perhaps not unexpected—considerable resemblance to Coulomb scattering.

Now we come to the correction terms to the usual eikonal approximation.[7] There are several variants of these corrections, all of which agree for small t. We shall exhibit the most straightforward (if not the most elegant) one and comment on another later. As we have stated the eikonal expansion is a perturbation series around $P=p$. With this in mind one may readily demonstrate that

$$\mathcal{T}(A) = \mathcal{T}_E(A) + AG_1(A)HG_1(A)A + AG_1(A)HG_1HG_1(A)A + \cdots, \tag{12}$$

with

$$H = -(P-p)^2, \tag{13}$$

and

$$G_1(A) = [2P\circ p - p^2 - m^2 - A(X) + i\epsilon]^{-1} = -i\int_0^\infty d\tau\, e^{i\tau[2P\cdot p-p^2-m^2]}\exp[-i\int_0^\tau d\sigma A(X-2p\sigma)]$$

$$= -i\int_0^\infty d\tau \exp[-i\int_{-\tau}^0 A(X-2p\sigma)d\sigma]\, e^{i\tau[2P\cdot p-p^2-m^2]}. \tag{14}$$

This corrected form of $\mathcal{T}(A)$ leads, via (5), to a corrected $\mathcal{T}(p_2', p_2; p_1', p_1)$. The first $\delta\mathcal{T}_E$ to the leading eikonal expression may be put in the form (on dropping terms which are negligible for t small)

$$\delta\mathcal{T}_E(p_2',p_2;p_1',p_1) = \int d^4y\, e^{i(p_1-p_2\cdot y)}\frac{d}{d\alpha}\Big[\{\exp[-\int_\alpha^\infty d\tau\int_0^\infty d\sigma D(y-2p\tau+2p'\sigma)]-1\}$$

$$\times\Box_y\{\exp[-\int_\alpha^\infty d\tau'\int_0^\infty d\sigma' D(y-2p\tau'+2p'\sigma')]-1\}$$

$$+\text{term with }(p_1\leftrightarrow p_1', p_2\to p_2')\Big]\Big|_{\alpha=0}. \tag{15}$$

Equation (15) is a fully relativistic version of the Saxon-Schiff[8] correction to the eikonal. If one wishes to use the eikonal expansion for large t he may proceed precisely along the path laid out by Sugar and Blankenbecler and make a symmetric expansion about both the initial and final directions. All their results, transcribed into four-dimensional language, go through. In particular one learns that if the "potential" falls as a power of t, then the two terms of the expansion are an excellent approximation for all t. If one were to imagine a generalized potential of the Regge form $s^{\alpha(t)}$ with the popular linear trajectories, then many terms of the expansion must be accounted for away from $t=0$.

From the point of view we have presented here it is clear why Regge behavior does not transpire. We have, in essence, a relativistic transcription of potential scattering for large s and t fixed. In potential theory Regge-like asymptotic forms come for large t and s fixed giving the usual $t^{\alpha(s)}$. There is without doubt a class of diagrams in the models we have considered here which yield t-channel ladders, straight and crossed, whose asymptotic behaviors surely sum up to $s^{\alpha(t)}$. Whether the eikonal graphs or the Regge graphs dominate has certainly not been determined here, nor until we learn to extract more cleanly the asymptotic behavior of the t-channel ladders will we be in a position to choose. As a speculation along these lines one might imagine that the Pomeranchuk or diffractive contribution to high-energy scattering comes from the eikonal graphs while the t-channel ladders yield the usual moving trajectories.

We happily acknowledge fruitful discussions with S. J. Chang, M. L. Goldberger, and S. Ma.

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*Research sponsored by the Air Force Office of Scientific Research, U.S. Air Force, under Contract No. AF49 (638)-1545.

†On leave from Départment de Physique Théorique, Centre d'Etudes Nucléaires de Saclay, 91-Gif-Sur-Yvette, France. Work supported by the National Science Foundation.

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PARTIAL-WIDTH FORMULATION OF UNITARITY SUM RULES FOR K_L-K_S DECAY

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(Received 25 April 1969)

By including K_S in the background which underlies K_L, S-matrix unitarity can be used to derive a set of constraints on the partial decay amplitudes of the two resonances. The only constraint which does not explicitly involve the strong-interaction background phases is equivalent to the Bell-Steinberger sum rule.

In 1965, Bell and Steinberger[1] derived a "unitarity sum rule,"

$$-i(M_L^*-M_S)\langle L|S\rangle=\sum\langle F|T|L\rangle^*\langle F|T|S\rangle, \quad (1)$$

satisfied by the amplitudes for the decay of K_L and K_S into final states F. Its practical significance is that it determines the phase of the right-hand side ($\sum$) in terms of $\langle L|S\rangle$, and $\langle L|S\rangle$ in turn is, e.g., real if CPT is conserved,[1] and imaginary if only T is conserved.[2] The sum rule is particularly noteworthy because it involves only the amplitudes for decay into the channel states F, and not the production amplitudes from the channels (2π, 3π, etc.), which are different if T is not conserved (and unobservable because they describe production via the weak interaction).

Although the Bell-Steinberger derivation involved only a consideration of the time dependence of the decay process, McGlinn and Polis[3] have recently suggested that by regarding K_L and K_S as conventional but overlapping resonances in the channels open at that energy, it should be possible to obtain an equivalent sum rule in terms of K-matrix partial-width amplitudes. They did so, but the sum rule they found in this way appeared to be different from the Bell-Steinberger result. The apparent difference seems to us to arise from a failure to distinguish between K-matrix and S-matrix decay amplitudes. To explain this, we have obtained yet a third sum rule, this one expressed in terms of S-matrix partial widths. We find that the phase information it contains is equivalent both to that of the Bell-Steinberger sum rule and to that of the McGlinn-Polis expression, so that in this sense all three results are equivalent.

Consider two overlapping resonances with the same quantum numbers. If in their neighborhood the energy dependence of the background can be neglected, the partial-wave S matrix for N open channels can be approximated by the two-pole