find a satisfactory answer to the following question: Should one parametrize the P_i, S, or perhaps both? Without some actual experience there is no definite answer at present. One reasonable exploratory program would be the following: Choose for S a delta-convergent sequence and parametrize it; then select a set of P_i that belong to some well-defined class of geometrical objects; finally optimize the parameters with respect to the primitive function φ_p. A very important example of geometrical objects is the class of all hypersurfaces. In particular, the integral of an N-dimensional function $F(\vec{x})$ over the hyperplane $x_1\mu_1 + \cdots + x_N\mu_N = p$ is called the Radon transform of $F(\vec{x})$.[9] Its general properties and geometrical meaning are well established.[9]

Assume now that the x_i, $i = 1, \cdots, L$, refer to the L electron-nucleus separation coordinates in a molecule. Using Eq. (2) we can construct L-center molecular orbitals that are completely different from the conventional linear combination of atomic orbitals ones and relate much more closely to the geometry of the molecule. Furthermore, the coalescence of the x_i produces a "united-atom" atomic orbital.

We have to discuss the questions of symmetry and statistics. The least sophisticated approach is to construct both the primitive function and the shape function in such a way that they are neither symmetry breaking nor statistics violating. Of course, this also imposes certain restrictions on the arguments of the P_i. Another method would consist of applying symmetrization (and antisymmetrization) operators as well as the appropriate projection operators at the end. The relative merits of these alternatives need further investiastion.

It has to be emphasized that Eq. (2) is not the most general correlated many-particle trial function one could imagine. First, the t_i need not be the scale factors. Second, the $P_i(\vec{t})$ could be made dependent also on the physical coordinates. (This would relate our functions to the more conventional collective-coordinate approach in nuclear physics.[10]) Finally, the general delta function we use could be replaced by an arbitrary function of $\vec{x}$ and $\vec{t}$. However, we feel that the formulation we propose combines conceptual simplicity and an appeal to geometrical intuition with the possibility of systematic classification of trial functions and computational practicability.

[1] R. L. Somorjai, Chem. Phys. Letters 2, 399 (1968).
[2] R. L. Somorjai, to be published.
[3] D. M. Bishop and R. L. Somorjai, to be published.
[4] R. L. Somorjai and D. M. Bishop, Phys. Rev. (to be published).
[5] D. D. Shillady, Chem. Phys. Letters 3, 17, 104 (1969).
[6] I. M. Gel'fand and G. E. Shilov, Generalized Functions (Academic Press, Inc., New York, 1964), Vol. I, p. 34.
[7] See Gel'fand and Shilov, Ref. 6, Chap. 3, p. 209.
[8] See Gel'fand and Shilov, Ref. 6, p. 348.
[9] I. M. Gel'fand, M. I. Graev, and N. Ya. Vilenkin, Generalized Functions (Academic Press, Inc., New York, 1966), Vol. V, Chap. 1.
[10] D. M. Brink, in Progress in Nuclear Physics, edited by O. R. Frisch (Pergamon Press, Inc., New York, 1960), Vol. 8, p. 97.

π-π PHASE SHIFTS IN THE REGION 1.0-1.4 GeV/c^2 *

B. Y. Oh,† W. D. Walker, J. T. Carroll, M. Firebaugh, A. Garfinkel,‡ and R. Morse
University of Wisconsin, Madison, Wisconsin 53706

and

J. D. Prentice, N. R. Steenberg, and E. West
University of Toronto, Toronto, Canada
(Received 24 March 1969)

$\pi\pi$ scattering in the di-pion mass region 1.0-1.4 GeV is analyzed. It is shown that an anomaly of the $\pi^-\pi^+$ state in the region 1.0-1.2 GeV is either an $I=0$ D-wave amplitude which interferes with a nearly static $I=1$ P-wave amplitude or a Breit-Wigner D wave which interferes with a moving P wave (possibly resonant). The f^0 meson seems to show considerable inelasticity.

An enhancement in the $\pi^-\pi^+$ mass spectrum in the di-pion mass region 1.0-1.2 GeV has been reported by Whitehead et al.,[1] Miller et al.,[2] and others.[3] It is noted in Refs. 1 and 2 that the observed enhancement[4] has $I^G = 0^+$ and J^P probably 2^+ and is not associated with the S^*, previously observed in the $K\bar{K}$ state.[5,6]

In this Letter we present an analysis of the enhancement in question from a study of the reactions

$$\pi^-p \to \pi^-\pi^+n \quad (4190 \text{ events}), \tag{1}$$

$$\pi^-p \to \pi^-\pi^0p \quad (3555 \text{ events}), \tag{2}$$

$$\pi^+d \to \pi^-\pi^+pp \quad (2636 \text{ events}), \tag{3}$$

at 7 GeV/c, and show that the anomaly of the $\pi^-\pi^+$ state in the region 1.1-1.2 GeV could be either an $I=0$ D-wave amplitude which interferes with a nearly static P-wave amplitude or a Breit-Wigner D wave which interferes with a moving P-wave (possibly resonant) amplitude.

The $\pi^-\pi^+$ enhancement in the region 1.1-1.2 GeV is seen to be correlated with a rapid variation of the $\cos\theta$ distribution with $M(\pi^-\pi^+)$ and a lack of such variation for the $\pi^-\pi^0$ state. This rapid variation has been known for four years or more.[7] Thus in Ref. 2, for example, the enhancement is seen only for the region of $\cos\theta < -0.75$ (θ is the $\pi^-\pi^-$ scattering angle in the $\pi\pi$ rest frame). A careful phase-shift analysis is needed for an understanding of the $\pi^-\pi^+$ anomaly, as apparently a number of partial-wave amplitudes of both parities seem to interfere strongly in the region 1.0-1.2 GeV.

We show in Figs. 1(a), 1(b), and 1(c) the $\pi^-\pi^+$ and $\pi^-\pi^0$ mass spectra of Reactions (1)-(3) (see below for curves). An excess of events is seen in the 1.1- to 1.2-GeV region of the $\pi^-\pi^+$ system above what would be expected for a simple Breit-Wigner f^0 meson. The signal is particularly enhanced in the region of $\cos\theta < -0.8$, as shown in Fig. 1(d) and as found in Ref. 2. The variation of the $\cos\theta$ distribution with the di-pion mass is shown in Fig. 2 for the $\pi^-\pi^+$ and $\pi^-\pi^0$ states of the Reactions (1) and (2). And in Figs. 1(e) and 1(f) are shown the corresponding $(F-B)/(F+B)$ ratio for the $\pi^-\pi^+$ and $\pi^-\pi^0$ states. The correlation of the 1.1- to 1.2-GeV anomaly with the $\cos\theta$ variation is clearly seen. The basic fact that one must consider is that the ratio $(F-B)/(F+B)$ becomes negative just above 1.0 GeV/c^2, and also, the anomaly in the mass spectrum shows mostly for $\cos\theta_{\pi\pi} < -0.8$. This tells one that the effect observed must be seen as the result of the interference of two states of opposite parity. It could be the result of S-P or P-D or S-P-D interference.

We believe that we are observing dominantly one-pion exchange (OPE) all through this mass region. We see the characteristic small momen-

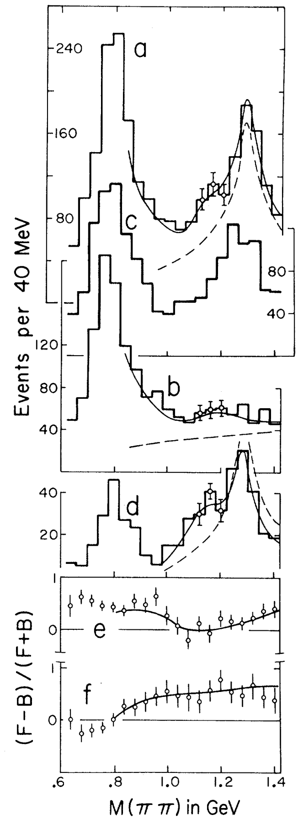

FIG. 1. (a), (c) $\pi^-\pi^+$ mass spectra from Reactions (1) and (3). (b) $\pi^-\pi^0$ mass spectrum from Reaction (2). (d) Mass spectrum of $\pi^-\pi^+$ in the region of $\cos\theta < -0.8$ from Reaction (1). (e), (f) Ratio $(F-B)/(F+B)$ of $\pi^-\pi^+$ and $\pi^-\pi^0$ versus di-pion mass for events of Reactions (1) and (2) with $-t < 0.3$ GeV$^2/c$. The solid curves correspond to the absolute cross sections of the AOPE model for the set of $\pi\pi$ phase shifts shown in Fig. 3(c): solution (d) in the text. The dashed curves correspond to the cross sections obtained when we leave out the P-wave amplitude completely while leaving everything else the same as for the solid curves.

tum transfers as we do in the ρ^0 and f^0 mass regions. The φ (Treiman-Yang) distributions are quite flat as would be expected from an OPE process. We have looked at the data of Crennell et al.[5] on the process $\pi^-p \to K_1^0K_1^0n$ and this process also seems to be consistent with OPE. The K^0K^0 angular distribution is quite consistent with being an S wave. The cross sections indicate that the S wave is very strongly absorbed in the process $\pi^-\pi^+ \to K\bar{K}$. Since the backward peak must come from the real part of the amplitude it is very difficult to see how S-P interference could possibly give the observed effect. The result is that the backward bump is almost certainly the result of P-D interference. The angular distributions are also consistent with this inter-

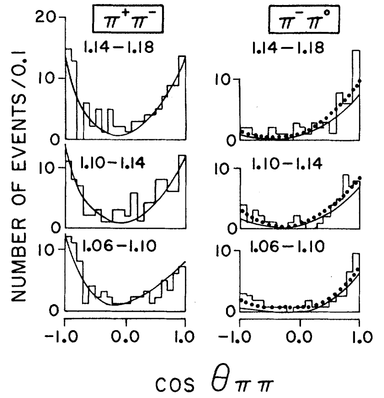

FIG. 2. $\cos\theta_{\pi\pi}$ for $\pi^+\pi^-$ and $\pi^-\pi^0$ from Reactions (1) and (2) in the mass region of interest. The solid and dotted curves (where they differ) correspond to solutions (a) and (d), as described in the text.

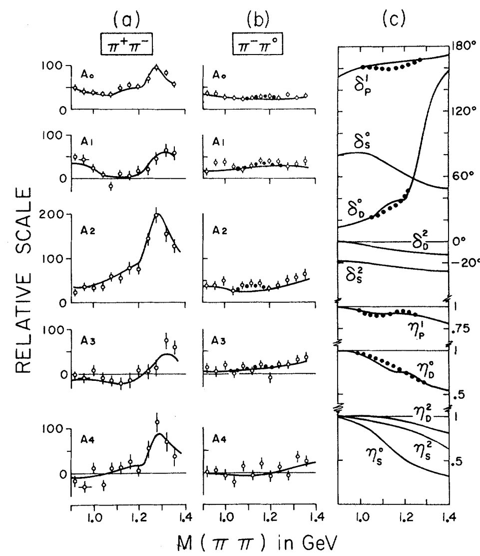

FIG. 3. (a), (b) Legendre coefficients in the expansion $d\sigma/d\cos\theta_{\pi\pi}dm_{\pi\pi}=\sum_l A_l P_l(\cos\theta_{\pi\pi})$ for Reactions (1) and (2) versus di-pion mass. The ordinate indicates the number of events for $dm_{\pi\pi}=0.04$ GeV. (c) The $\pi\pi$ phase shifts in the expression $A_l^I=[\eta_l^I \times\exp(2i\delta_l^I)-1]/2ik$ vs $m_{\pi\pi}$. The solid and dotted curves (where they differ) correspond to solutions (a) and (d), respectively, as described in the text.

pretation. The backward hemisphere is always more sharply peaked than the forward hemisphere for the $\pi^-\pi^+$ state in the mass region 1.05-1.2 GeV/c^2 as may be seen in Fig. 2. The same result is observed from examining coefficients of the Legendre expansion of the angular distribution in Figs. 3(a) and 3(b). The negative A_3 coefficient indicates P-D interference.

We use the formalism of absorption-modified one-pion–exchange model (AOPE)[8,9] to write the differential cross section for the reactions. In applying the AOPE model for the region 1.1-1.2 GeV we note the following features: (1) We have to consider five partial-wave amplitudes, i.e., A_S^0, A_S^2, A_P^1, A_D^0, and A_D^2 (neglecting $l\geq 3$ waves), all of which may be partially inelastic. (2) From the $\cos\theta$ distribution of the $\pi^-\pi^0$ state and from the expected behavior of A_P^1 which has passed resonance ($\delta_P^1>90°$) the phase shift δ_D^2 should be small and negative. δ_S^2 is also expected to be small and negative if we assume no rapid variation of δ_S^2 from its behavior below 1 GeV.[1] Independent evidence in agreement with these assumptions about the small phase shifts for the $I=2$ waves comes from a study of reactions $\pi^-p\to\Delta^{++}\pi^-\pi^-$ and $\pi^+p\to\pi^+\pi^+n$.[10] (3) δ_D^0 is expected to be positive, growing rapidly to reach 90° at the f^0 meson peak. (4) The $I=0$ S-wave amplitude is an open question. The difficulty is mainly due to its low unitarity bound which makes the results insensitive to the choice of the S-wave amplitude. (5) The amplitude vectors in the Argand diagram of the important opposite-parity waves must be widely separated to give the observed negative $(F-B)/(F+B)$ ratio of the $\pi^-\pi^+$ state.

With the above boundary conditions on the A_l^I's we have tried a great number of possible sets of phase shifts to fit the observed cross section $\partial^3\sigma/\partial m_{\pi\pi}\partial\cos\theta\partial\varphi$. We have tested in particular the hypotheses of (a) D-wave resonance (loop or cusp) with other waves varying slowly, (b) an $I=0$ S-wave resonance rapidly sweeping the unitarity circle with other waves varying slowly (δ_D^0 varying according to a Breit-Wigner formula), (c) a constant P-wave phase shift of 150°-155° and $\eta_P^1=0.8$-1.0 with other waves varying slowly (δ_D^0 varying according to a Breit-Wigner formula), and (d) a resonant P wave (loop) near the phase shifts given in (c) above.

Hypothesis (b) gives no solution; i.e., the observed anomaly is too large an effect to be caused by an S-wave amplitude, consistent with the observations of Refs. 1 and 2. Hypothesis (a) cannot be ruled out ($\chi^2=92$ with 110 degrees of freedom). Hypothesis (d) also gives good agreement with the data ($\chi^2=82$ with 110 degrees of freedom). One cannot really decide between these two possibilities or a combination of them (small loops in the Argard diagram for both P and D waves) on the basis of this experiment alone. A recent experiment of Armenise et al.[11] has looked at the missing-mass spectrum in the process

$\pi^+d \to mm^0 + pp_s$. This spectrum and preliminary results from an experiment we have done indicate structure in the $2\pi^0$ state in the 1.1-GeV/c^2 mass range. This would indicate an effect in the $I=0$ state and thus favor hypothesis (a).

The sets of phase shifts for solutions (a) and (d) are shown in Fig. 3(c) as solid and dotted curves (where the two solutions differ), respectively. The predictions of these solutions are shown as the smooth curves on Figs. 1, 2, and 3 [again, where the predictions differ, solution (d) is shown as the dashed curve]. We have also shown in dashed curves on Figs. 1(a), 1(b), and 1(d) the di-pion mass distributions which are expected when we leave out the P-wave amplitude completely while leaving everything else the same as for the solid curves. Thus the P-wave amplitude, which accounts for the difference between the solid and dashed curves, is quite evident in the region 1.1-1.2 GeV. An apparent lack of a bump structure for the $\pi^-\pi^0$ state is seen to be due to an almost constant magnitude of the elastic P-wave amplitude in the region 1.1-1.2 GeV. If the P-wave amplitude is indeed resonant, making a small loop in the Argand diagram, its mass and width appear to be 1120^{+100}_{-40} MeV and 150^{+100}_{-40} MeV, respectively. This agrees with the mass value predicted by Barger and Phillips[12] for the ρ' from a Regge-pole analysis.

The extremely small change in the P-wave amplitude in this mass range would indicate a very small branching ratio of such a resonance into the 2π state and thus would not be seen easily in photoproduction experiments detecting pairs of pions. The inelastic P-wave scattering may contribute in part to the $\pi^-\omega$ enhancement (probably apart from the B meson) in the region 1.1-1.3 GeV, as has been suggested by Parkinson,[13] or to K pair production.[14] In any event the existence of a resonance, while of great current interest,[15] would have to be verified by observation in an inelastic channel. We should also note that the f^0 seems to show considerable inelasticity in the region of resonance.[16]

We wish to thank our measurers and scanners, and in particular, Mrs. L. Boyd and Mrs. S. Maggs for their aid and cooperation in this work. We wish also to thank Professor L. Durand and Professor V. Barger for their discussions of the data. The cooperation of the 30-in. bubble-chamber crew and Dr. L. Voyvodic were essential for the success of the work.

*Work supported in part by the U. S. Atomic Energy Commission under Contract No. AT(11-1)-881, COO-881-211, and by the National Research Council of Canada.

†Present address: Physics Department, Michigan State University, East Lansing, Mich.

‡Present address: Physics Department, Purdue University, Lafayette, Ind.

[1]C. Whitehead, J. G. McEwen, R. J. Ott, D. X. Aitken, G. Bennett, and R. E. Jennings, Nuovo Cimento 53A, 817 (1967).

[2]D. H. Miller, L. J. Gutay, P. B. Johnson, V. P. Kenny, and Z. G. T. Guiragossián, Phys. Rev. Letters 21, 1489 (1968).

[3]Wisconsin-Toronto Collaboration, unpublished; B. Y. Oh, thesis, University of Wisconsin, 1968 (unpublished).

[4]The mass and width of the enhancement reported in Refs. 1 and 2 are $M = 1085$ MeV and $\Gamma < 25$ MeV, and $M = 1060 \pm 15$ MeV and $\Gamma < 70$ MeV, respectively. Quantum numbers are deduced in both Refs. 1 and 2 from unitarity arguments and nonobservation of enhancement in the $\pi^-\pi^0$ state, or lack of strong P- and D-wave interference. $J^P = 0^+$ is not completely ruled out in Ref. 1.

[5]D. J. Crennell, G. R. Kalbfleisch, K. W. Lai, J. M. Scarr, T. G. Schumann, I. O. Skillicorn, and M. S. Webster, Phys. Rev. Letters 16, 1025 (1966).

[6]W. Beusch, W. E. Fischer, B. Gobbi, M. Pepin, E. Polgar, P. Astbury, G. Brautti, G. Finocchiaro, J. C. Lassale, A. Michelini, K. M. Terwilliger, D. Websdale, and C. H. West, Phys. Letters 25B, 357 (1967).

[7]Aachen-Birmingham-Bonn-Hamburg-London-München Collaboration, Phys. Letters 5, 153 (1963); see also I. Ohba and T. Kobayashi, Progr. Theoret. Phys. (Kyoto), Suppl. Extra No. 1967, 90.

[8]K. Gottfried and J. D. Jackson, Nuovo Cimento 34A, 735 (1964); L. Durand and Y. T. Chiu, Phys. Rev. 139, B646 (1965).

[9]We use the formalism of Durand and Chiu (Ref. 8). The S and P amplitudes are given by Durand and Chiu and the $(\pi\pi)$ D-wave amplitudes are similarly derived (Ref. 3). We have also included in our amplitudes the effect of the nucleon pole terms. These terms have a small effect as they tend to cancel for the $\pi\pi$ mass range under consideration. See E. West et al., Phys. Rev. 149, 1089 (1966), and Ref. 3 for a discussion of nucleon pole terms.

[10]See, for example, I. Butterworth's review, in Proceedings of the International Conference on Elementary Particles, Heidelberg, Germany, 1967, edited by H. Filthuth (North-Holland Publishing Company, Amsterdam, The Netherlands, 1968); or Genova-Hamburg-Milano-Saclay, Nuovo Cimento 57A, 699 (1968), and our own data [R. Morse, University of Wisconsin, thesis, 1969 (unpublished)].

[11]N. Armenise et al., Nuovo Cimento 54A, 999 (1968).

[12]V. Barger and R. J. N. Phillips, Phys. Rev. Letters 21, 865 (1968).

[13]M. Parkinson, Phys. Rev. Letters 18, 270 (1967).

[14]M. Aguilar-Benitez et al., Phys. Letters 29B, 241 (1969). These authors give evidence that the S^* is a 2^+

object. While this conclusion is consistent with the inelasticity we observe for the $I=0$ D wave, we note that it depends strongly on successfully eliminating the $K^*(890)$ contribution.

[15]G. Veneziano, Nuovo Cimento 57A, 190 (1968); C. Lovelace, Phys. Letters 28B, 265 (1968).

[16]See, G. Ascoli, H. B. Crawley, D. W. Mortara, and A. Shapiro, Phys. Rev. Letters 21, 1712 (1968), for the 4π decay of the f^0, and Ref. 3.

$\pi^0\pi^0$ MASS SPECTRUM AND δ_0^0 BELOW 1 GeV *

Gerald A. Smith
Physics Department, Michigan State University, East Lansing, Michigan 48823

and

Robert J. Manning
Lawrence Radiation Laboratory, Berkeley, California 94720
(Received 18 June 1969)

We have measured the δ_0^0 phase shifts from $\pi^+\pi^-$ elastic scattering and compared the four possible solutions with our $\pi^0\pi^0$ mass spectrum. Only one solution, characterized by a very gradual increase of the phase shifts to 90°, compares favorably.

In this paper we present a unique set of $I=0$, $J=0$ π-π scattering phase shifts (δ_0^0) based on an analysis of the reactions

$$\pi^+n \to p\pi^+\pi^- \tag{1}$$

and

$$\pi^+n \to p\pi^0\pi^0 \tag{2}$$

near 2-GeV/c incident-pion momentum. This paper differs from an earlier report[1] in essentially two respects: (a) We now include in the analysis a new sample of 4044 low-momentum-transfer events from Reaction (1), thus providing a prediction for the $\pi^0\pi^0$ mass spectrum within the framework of our own experiment, and (b) we have redone background subtractions on the $\pi^+n \to p+$"neutrals" events based on considerably more reliable information for the $p+3\pi^0$ reaction. We find that only one set of phase shifts, the so-called up-down solution, satisfies both the $\pi^+\pi^-$ and the $\pi^0\pi^0$ data. This solution is characterized by a gradual increase of the phase shifts to 90° near 750 MeV, thereafter remaining constant to 1000 MeV.

The method described by Schlein[2] and applied to π-π scattering by Malamud and Schlein[3] has been adopted for the phase-shift analysis. Since the $I=2$, $J=0$ phase shifts are not available to us in this experiment as a consequence of the fact that a deuterium target has been used, we have assumed values of δ_0^2 based on averaging values reported by several authors.[4] Since the method of analysis has been discussed elsewhere, only the relevant points will be reviewed here. We have used the coordinate definitions and expressions given by Schlein for the moments $N\langle Y_l^m\rangle$ and cross sections of the π-π scattering angular distributions in order to fit our experimental data. The quantities $|\vec{s}|^2$, $|\vec{p}_1-\vec{p}_{-1}|^2$, $|\vec{p}_0|^2$, and $(|\vec{p}_1|^2+|\vec{p}_{-1}|^2)$ are helicity-amplitude quantities, and $\theta(\vec{p}_0,\vec{s})$ and $\theta(\vec{s},\vec{p}_1-\vec{p}_{-1})$ are angles between the helicity amplitudes. All off-mass-shell and absorption corrections to the simple one-pion-exchange (OPE) Born helicity amplitudes are included in the definition of the above helicity-amplitude quantitites. Expressions for the S- and P-wave scattering amplitudes involving explicitly the δ_0^0, δ_1^1, and δ_0^2 phase shifts may be found in Ref. 3.[5,6]

A crucial assumption of this method of analysis is that the entire π-π mass dependence of the moment equations resides in the π-π scattering amplitudes which are assumed to be independent of t and E^* (overall c.m. energy). Likewise, the helicity-amplitude quantities are assumed to depend only on t and E^*. A recent article by Bander, Shaw, and Fulco[7] has shown that below a π-π mass of 600 MeV this assumption may not be valid. Since in our analysis we have examined regions of the Chew-Low plot down to 500 MeV, it is necessary to check this assumption. We have evaluated the moments of the π-π scattering angular distributions as a function of π-π mass for several limits on t. We have found that the inclusion of data between 500 and 600 MeV does not significantly raise or lower the confidence level for either the $l=1$ or 2 moment tests. Therefore, we feel that the initial assumption on the constancy of the helicity-amplitude quantities as a function of π-π mass is valid down to 500