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BROKEN SYMMETRY AND DECAY OF ORDER IN RESTRICTED DIMENSIONALITY

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The ordering of one- and two-dimensional systems with a continuous symmetry is considered in the absence of a symmetry-breaking field. It is shown rigorously that no spontaneous ordering can occur; bounds on the order-order correlation function integrated over a subdomain indicate how the short-range order decays with distance.

It has been appreciated heuristically for some time that in a one- or two-dimensional system, i.e., a system of finite cross section or thickness, which has a continuous symmetry (such as the gauge invariance of a Bose fluid or rotational isotropy in a ferromagnet), the fluctuations in the order parameter are so large as to destroy any ordered state with spontaneously broken symmetry even though such can arise in the fully three-dimensional system. Hohenberg[1] has demonstrated that Bogoliubov's inequality,

$$\tfrac{1}{2}\langle\{A, A^\dagger\}\rangle \geq k_B T |\langle[C, A]\rangle|^2 / \langle[[C, \mathcal{H}_\Omega], C^\dagger]]\rangle, \quad (1)$$

in which $\mathcal{H}_\Omega$ is the Hamiltonian for the system confined to a domain Ω, can be used to substantiate this idea, and Mermin and Wagner[2] have proven that if the dimensionality of Ω is less than three, an isotropic Heisenberg ferromagnet can exhibit no spontaneous magnetization, i.e.,[3]

$$M_0(T) = \lim_{H\to 0^+} M(T, H) = 0 \quad (T > 0). \quad (2)$$

As indicated by (2) [see also Chester, Fisher, and Mermin,[4]] the existing proofs[5] first introduce a symmetry-breaking field η (the magnetic field H for a ferromagnet), then proceed to the thermodynamic limit [volume $V(\Omega)\to\infty$], and finally, show that the induced order parameter $\Psi(\eta)$ vanishes as the field η is removed ($|\eta|\to 0$). For a magnet $\Psi \sim M$, while for a Bose fluid one considers

$$\Psi(T, \eta) = \lim_{V(\Omega)\to\infty} [V(\Omega)]^{-1} \int_\Omega \langle\psi(\vec{r})\rangle_\Omega d\vec{r}. \quad (3)$$

These results are satisfying, but they leave open some more fundamental questions, namely:

(A) How does the order-order correlation function $\sigma(\vec{r}, \vec{r}\,')$ behave as $|\vec{r}-\vec{r}\,'|\to\infty$? For a magnet with localized spin variables $\vec{S}(\vec{r})$ we may take

$$\sigma(\vec{r}, \vec{r}\,') = \langle S_z(\vec{r}) S_z(\vec{r}\,')\rangle \text{ or } \langle S_+(\vec{r}) S_-(\vec{r}\,')\rangle. \quad (4)$$

One would like to say something about the rate of decay and to prove that $\sigma\to 0$ as $|\vec{r}-\vec{r}\,'|\to\infty$, so as to demonstrate the absence of long-range order [$\sigma(\infty)\equiv 0$]; but as a matter of fact, even when (2) holds one cannot be sure that $\sigma(\infty) = 0$.[6] For a Bose fluid one is interested in the off-diagonal order or one-body density matrix

$$\sigma(\vec{r}, \vec{r}\,') = \langle\psi^\dagger(\vec{r}\,')\psi(\vec{r})\rangle. \quad (5)$$

A second question is the following:

(B) Can one dispense with the symmetry-breaking field in proving the absence of ordering? (This question is especially pertinent for a Bose fluid,[7] where the relevant "off-diagonal" field cannot be realized physically.) An answer might be provided by considering (with $\eta\equiv 0$) the rms order parameter Ψ_σ defined by

$$(\Psi_\sigma)^2 = \lim_{V(\Omega)\to\infty} [V(\Omega)]^{-2} \times \int_\Omega d\vec{r} \int_\Omega d\vec{r}\,' \sigma_\Omega(\vec{r}, \vec{r}\,'), \quad (6)$$

where, as above, the subscript Ω indicates that the finite system is implied. One normally expects that

$$\Psi_0 \equiv \lim_{\eta\to 0} \Psi(\eta) = c\Psi_\sigma,$$

which would be of order unity for $d=3$, but this has never been shown generally. [Here c is a constant depending on the symmetry group and the precise definition of $\sigma(\vec{r}, \vec{r}\,')$.]

In this note we present, we believe for the first time, rigorous answers to these questions. Specifically we have answered (B) by proving that Ψ_σ vanishes for all $T>0$ if the domain Ω can be contained in a cylinder of finite cross section ($d=1$) or between parallel planes of finite separation ($d=2$). As regards (A) we prove that for $\eta=0$ and any (reasonably shaped) subdomain $\Gamma\subset\Omega$ which constitutes a "slice" of Ω, as shown in Fig. 1, we have the bounds

$$\Psi\{f\,|\,\Gamma\} \le \mathrm{const}\times[\ln V(\Gamma)]^{1/2}, \quad d=2,$$
$$\le \mathrm{const}\times[V(\Gamma)]^{-1/4}, \quad d=1, \tag{7}$$

as $V(\Gamma)\to\infty$, where

$$[V(\Gamma)\Psi\{f\,|\,\Gamma\}]^2 = V(\Gamma)n\{f\} = \int_\Gamma d\vec{r} \int_\Gamma d\vec{r}\,' f^*(\vec{r}\,')f(\vec{r})\sigma(\vec{r}, \vec{r}\,') \tag{8}$$

in which $|f(\vec{r})|=1$. [In a lattice system sums over cells replace the integrals.] This proves that there can be no (short) long-range order.[6] Roughly speaking it also shows that $\sigma(\vec{r}, \vec{r}\,')$ must decrease faster than $1/\ln|\vec{r}-\vec{r}\,'|$ for $d=2$ or $1/|\vec{r}-\vec{r}\,'|^{1/2}$ for $d=1$. [This does not insure that $\sigma(\vec{0}, \vec{r}\,')$ is integrable; thus "weak long-range order,"[4] or an infinite "susceptibility," could still arise.] If Ω and Γ have sufficiently regular shapes that $f(\vec{r}) = e^{i\vec{K}\cdot\vec{r}}$ can be regarded as a "single-particle state" of a Bose system, then (7) asserts that there can be no macroscopic occupancy of the state $\vec{K}$, i.e., $n_{\vec{K}}/N\to 0$, where $N = N(\Gamma)$ is the (mean) number of particles in Γ. Finally we note that (7) remains valid even if a symmetry-breaking field is imposed anywhere outside the subdomain Γ.

We now sketch the proof of (7) for a two-dimensional Bose system (the $d=1$ case is similar); the proofs for magnetic systems are somewhat simpler as are the proofs that $\Psi_\sigma \equiv 0$. These proofs, and further details, will be published separately. We suppose that the slice subdomain Γ is surrounded by a corridor or channel Δ, of thickness say between b and $2b$. When $V(\Gamma)\to\infty$ we assume that $V(\Delta)/V(\Gamma)$ approaches zero as a normal surface to volume ratio. Bogoliubov's inequality is now applied with

$$C = \int d\vec{r}\, g(\vec{r})\rho(\vec{r}) = \int d\vec{r}\, g(\vec{r})\psi^\dagger(\vec{r})\psi(\vec{r}), \tag{9}$$

which is the normal choice except for the weighting factor $g(\vec{r}) = a(\vec{r})\exp[i\vec{K}\cdot\vec{r}]$ (k arbitrary), which we suppose vanishes identically outside $\Gamma\cup\Delta$. We take $a(\vec{r})$ twice continuously differentiable with $a(\vec{r})=1$ for $\vec{r}$ in Γ, and $|\nabla a|\le 1/b$ in Δ, where $a(\vec{r})$ goes smoothly to zero. It is assumed that $\mathcal{H}_\Omega$ contains (i) a kinetic-energy term, (ii) only normal "diagonal" many-particle interactions, and (iii) a wall potential which ensures that all wave functions vanish smoothly on the boundary of Ω. We then obtain ($\hbar=1$)

$$\langle[[C,\mathcal{H}_\Omega], C^\dagger]\rangle = m^{-1}\int d\vec{r}\,|\nabla g|^2\langle\rho(\vec{r})\rangle$$
$$\le m^{-1}N(\Gamma\cup\Delta)[k^2+\lambda], \tag{10}$$

where the second line follows by noting that $|\nabla g|^2$ reduces to k^2 in Γ but in Δ becomes $k^2+|\nabla a|^2$, which is bounded by k^2+b^{-2}. Thus the parameter $\lambda = (1/b)^2 N(\Delta)/N(\Gamma\cup\Delta)$ grows small as $V(\Gamma)\to\infty$. Next we choose[7]

$$A = \int d\vec{r} \int d\vec{R}\, f^*(\vec{r})e^{-i\vec{K}\cdot\vec{r}} f(\vec{R})\psi^\dagger(\vec{r})\psi(\vec{R}), \tag{11}$$

where $|f(\vec{r})|$ is unity for $\vec{r}$ in Γ but vanishes otherwise. Then the numerator in (1) becomes

$$|\langle[C,A]\rangle|^2 = [V(\Gamma)(n\{f\} - n\{fe^{i\vec{k}\cdot\vec{r}}\})]^2, \tag{12}$$

where $n\{f\}$ is defined in (8). If D is the spacing between the parallel planes containing Ω, we have[8] for all $\vec{r}$ and $\vec{r}\,'$ in Ω (and Γ)

$$D^{-1}\sum_{\vec{k}_\perp}(2\pi)^{-2}\int d^2\vec{k}_\parallel\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')} = \delta(\vec{r}-\vec{r}\,'), \tag{13}$$

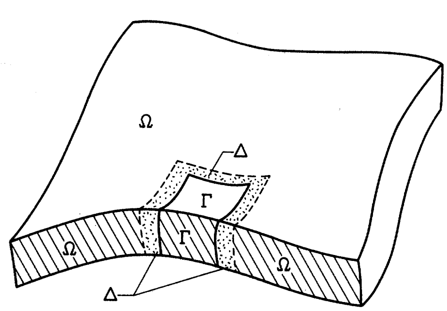

FIG. 1. Sectioned "two-dimensional" domain Ω showing a "slice" subdomain Γ and a surrounding corridor Δ.

where $\vec{k}=(\vec{k}_{\parallel}, \vec{k}_{\perp})$ in which $\vec{k}_{\perp}$ represent a discrete but complete set (including $\vec{k}_{\perp}=0$) of wave vectors normal to the bounding planes. Now we integrate the inequality for $|\vec{k}_{\parallel}| \leqslant \kappa$ with $\vec{k}_{\perp}\equiv 0$.[9] We may extend the integral (and sum) to all $\vec{k}$ over nonpositive terms on the right-hand side and nonnegative terms on the left-hand side. For the right-hand side we find, dropping a positive (squared) term,

$$R \geqslant (mk_BT)[V(\Gamma)^2/N(\Gamma \cup \Delta)] \times [n^2\{f\}I(\lambda)-2n\{f\}J(\lambda)], \quad (14)$$

where, denoting the restricted integration by subscript κ,

$$I(\lambda)=(2\pi)^{-2}\int_\kappa d^2\vec{k}/(k^2+\lambda) \approx (4\pi)^{-1}\ln(\kappa^2/\lambda) \quad \text{as } \lambda\to 0, \quad (15)$$

and

$$0 \leqslant J(\lambda) = (2\pi)^{-2}\int_\kappa d^2\vec{k}\, n\{fe^{i\vec{k}\cdot\vec{r}}\}/(k^2+\lambda) \leqslant \lambda^{-1}(2\pi)^{-2}\int_\kappa d^2\vec{k}\, n\{fe^{i\vec{k}\cdot\vec{r}}\} \leqslant \lambda^{-1}DN(\Gamma)/V(\Gamma)=\lambda^{-1}D\rho(\Gamma), \quad (16)$$

where (i) the positivity of n was used, (ii) the integration was extended to all $\vec{k}$, and (iii) relation (13) was applied. On the left-hand side we write $\langle\{A^\dagger, A\}\rangle = 2\langle AA^\dagger\rangle + \langle[A^\dagger, A]\rangle$ and extend the integral on the first term (only). On using the commutation relations and discarding appropriate negative terms we find

$$L \leqslant D[Q\{f\}+\rho(\Gamma)V(\Gamma)^2 - V(\Gamma)n\{f\}] + c_2V(\Gamma)^2\kappa^2 n\{f\}, \quad (17)$$

where c_2 is a constant and

$$0 \leqslant Q\{f\} = \int_\Gamma d\vec{R}\int_\Gamma d\vec{r}\int_\Gamma d\vec{r}\,' f^*(\vec{r})f(\vec{r}\,') \times \langle\rho(\vec{R})\psi^\dagger(\vec{r})\psi(\vec{r}\,')\rangle. \quad (18)$$

If the number of particles in Γ were fixed (definite), this would reduce simply to $N(\Gamma)V(\Gamma)n\{f\}$. However, we can allow for the natural fluctuations by using the relation[10]

$$\int_\Gamma d\vec{r}\int_\Gamma d\vec{r}\,'[\langle\rho(\vec{r})\rho(\vec{r}\,')\rangle - \langle\rho(\vec{r})\rangle\langle\rho(\vec{r}\,')\rangle] = k_BT\rho(\Gamma)^2V(\Gamma)K_T[1+\epsilon(\Gamma)], \quad (19)$$

where K_T is the isothermal bulk compressibility of the fluid in Γ which we may assume is bounded, and $\epsilon(\Gamma)\to 0$ as $V(\Gamma)\to\infty$ represents a surface-to-volume correction. Application of Schwarz's inequality to (18) then yields

$$Q\{f\} \leqslant c_3\rho(\Gamma)^2[k_BTK_T(1+\epsilon)]^{1/2}V(\Gamma)^{5/2} + \rho(\Gamma)V(\Gamma)^2n\{f\}, \quad (20)$$

where c_3 is a constant

Finally on collecting terms and using the definition (8) the inequality reduces to the form

$$q_1\Psi^2 + q_2[V(\Gamma)]^{-1/2} \geqslant \Psi^4 I(\lambda) \sim \Psi^4 \ln V(\Gamma), \quad (21)$$

where q_1 and q_2 are intensive parameters depending on temperature and density. On multiplying by $I(\lambda)$ and choosing $V(\Gamma)$ so large that $\ln V(\Gamma)/V(\Gamma)^{1/2} \ll 1$ we obtain the desired result (7).

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[2] N. D. Mermin and H. Wagner, Phys. Rev. Letters 17, 1133 (1966).

[3] Here $M_0(T)$ and $M(T,H)$ denote the magnetizations per spin (or per unit volume) at temperature T and external field H of an infinite system (see below).

[4] G. V. Chester, M. E. Fisher, and N. D. Mermin, Phys. Rev. (to be published).

[5] Similar proofs have been presented for crystalline ordering, other sorts of magnetic ordering, etc., e.g., N. D. Mermin, J. Math. Phys. 8, 1061 (1967), and Phys. Rev. 176, 250 (1968).

[6] The point is that $\sigma(\vec{r}, \vec{r}\,')$ is defined by taking the thermodynamic limit with fixed $\vec{r}$ and $\vec{r}\,'$; thus $\sigma(\infty)$ is then the "short long-range order" in contrast to the "long long-range order" σ_∞ in which $|r-r'|$ is, say, kept equal to $[V(\Omega)]^{1/d}$ as the thermodynamic limit is taken.

[7] A discussion of this question has been presented by G. V. Chester [Lectures in Theoretical Physics (University of Colorado Press, Boulder, Colo., to be published), Vol. XI] and our work follows his in spirit.

[8] Following Chester, Fisher, and Mermin, Ref. 4.

[9] To prove Ψ_σ vanishes we also employ a lower limit $\kappa_0 < |\vec{k}_{\parallel}|$ which then plays a similar role to λ.

[10] This is merely a form of the standard (compressibility)/(fluctuation) relation. Although we are not aware of a rigorous general proof, we accept it because, here, we need to assume only that (19) is valid with some constant K_T. This simply embodies the physical assumption that the density fluctuations are "normal."