Z DEPENDENCE OF RADIATIVE CORRECTIONS TO β DECAY

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We show that to first order in α, and with neglect of effects which vanish in the limit of zero lepton momenta, the theoretical ft value in $0 \rightarrow 0$ Fermi transitions is independent of Z. The only model-dependent feature of this argument is in the treatment of effects induced by the axial current. It is further shown that in higher order radiative corrections the power of Z cannot exceed the power of α; the leading term in the radiative correction to the ft value is therefore of order $Z^2\alpha^2$.

The near equality of the ft values in the nine $0^P \rightarrow 0^P$ ($P=+$ or $-$) superallowed Fermi transitions[1] is generally regarded as one of the more spectacular successes of the conserved-vector-current (CVC) hypothesis of Feynman and Gell-Mann, and Gerstein and Zeldovich.[2] Observe, however, that the electric charges of the decaying states range from $Z=1$ for π^+ to $Z=27$ for Co. Since electromagnetism destroys the conservation of the isospin current, why does the ft value for Co^{54} continue to be nearly equal to that for π^+? A naive perturbation expansion of the radiative corrections to the "bare" element would in fact be a power series not in α ($\approx 1/137$) but the uncomfortably large number $Z^2\alpha$ (≈ 5 for Co) or more generally $Z^m\alpha^n$. The purpose of this note is to report on a study of the Z dependence of radiative corrections to β decay, within the framework of field theory, which avoids as much as possible specific nuclear models and which is valid to all orders in α. Our main results can be states succinctly in the form of two theorems:

Theorem I: If CVC is broken only by electromagnetism, then (a) to first order in α, (b) to zeroth order in the lepton momenta,[3] and (c) with neglect of induced effects stemming from the axial-vector current, the ft value is independent of Z.

Theorem II: The Z-dependent renormalization effects, which do appear when one goes to higher orders in the electric charge, are such that any potential power of Z is matched—or exceeded—by a power of α. That is to say, a renormalization induced by n virtual photons is at most of order $(Z\alpha)^n$, not $(Z^2\alpha)^n$ as a naive counting might indicate.

These theorems, proved below, enable us to assert that the Z-dependent radiative corrections to the ft values are at worst of order $Z^2\alpha^2$; barring accidental dynamical enhancements, this is no worse than 0.35% for O^{14} and 3.9% for Co^{54}.

One-photon effects.—These are conveniently discussed by reference to the six graphs of Fig. 1. We have split the radiative effects into six pieces in order to display explicitly the isotopic transformation property of the hadronic current coupled to the electromagnetic field. All external lines are endowed with physical masses. Mass counter terms are therefore required to ensure that the masses stay at their physical val-

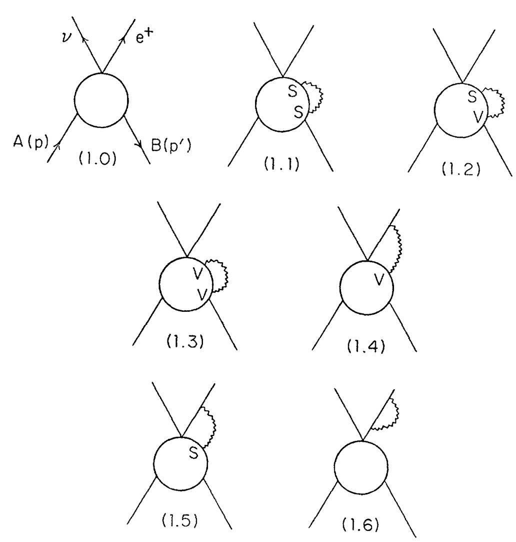

FIG. 1. Radiative corrections to β decay to order α.

ue in the presence of the electromagnetic field; such terms are understood but not displayed.

Following the standard practice in electrodynamics with minimal coupling, we ascribe to each s-type vertex a coupling strength proportional to the isoscalar charge of the system ($=\frac{1}{2}eA \approx Ze$, for all the heavier nuclei of Ref. 1); to each v-type vertex we ascribe a coupling strength e (all the nuclei of Ref. 1 have $|I_3| \leq 1$).

We note first that the matrix element $\langle B|I_+|A\rangle$ which appears in Fig. (1.0) is not strictly $\sqrt{2}$; the point is that if the states A and B have physical masses, they are not eigenstates of isospin. There is a standard theorem,[4] however, which assures us that the deviation of this matrix element from $\sqrt{2}$ is of second order in the mass splitting, i.e., $O(Z^2\alpha^2)$. The mass splitting can, of course, be ignored in Figs. (1.1)-(1.6), since we are working only to order α.

The contributions from Figs. (1.1)-(1.3) can be written as ($i,j = s$ or v; no summation over repeated i or j)

$$M^{ij} = -\frac{\alpha}{8\pi^3}\frac{G_V}{\sqrt{2}}(1-\tfrac{1}{2}\delta_{ij})L_\lambda \int \frac{d^4k}{k^2+ie} g^{\mu\nu}\frac{\partial}{\partial k_\lambda}\int d^4y\, e^{-ik\cdot y}\langle B|T\{[I_+, J_\mu{}^i(0)]J_\nu{}^j + i \rightarrow j\}|A\rangle. \tag{1}$$

In Eq. (1), L_λ is the lepton current and J^S (J^V) the isoscalar (isovector) part of the hadronic electromagnetic current, and the T product is taken to be "covariantized" in the usual way.[5] The derivation of Eq. (1) involves straightforward and well-known manipulations with Ward-Takahashi identities and need not be reproduced here.[6]

The order of magnitude of the contributions from the various graphs can now be stated [notation: $M(1.1) \equiv$ contribution to amplitude from Fig. (1.1)].

$M(1.1) \equiv M^{ss}$: potentially of order $Z^2\alpha$, actually zero, as is obvious from Eq. (1).

$M(1.2) \equiv M^{sv}$: potentially of order $Z\alpha$, actually zero by virtue of the TP invariance of strong and electromagnetic interactions. [TP implies that, in this case, integration over y in Eq. (1) leads to an even function of k.]

$M(1.3) \equiv M^{vv}$: $O(\alpha)$.

$M(1.4)$: $O(\alpha)$.

$M(1.5)$: $O(Z\alpha)$.

$M(1.6)$: $O(\alpha)$.

The contribution to $M(1.5)$ from the vector current may be written as

$$M(1.5)_V = \frac{-i\alpha}{2\pi^3}\frac{G_V}{\sqrt{2}}L^\rho l^\mu \int \frac{d^4k}{(k^2+i\epsilon)(k^2+2l\cdot k+i\epsilon)}\left\{g_{\mu\lambda}g_{\rho\nu} + \frac{k_\rho k_\mu g_{\lambda\nu}}{k^2+i\epsilon}\right\}V^{\lambda\nu}, \tag{2}$$

where

$$V_{\lambda\nu} = i\int d^4x\, e^{-ik\cdot x}\langle B|T\{J_\lambda{}^S(x)V_\nu{}^{(+)}(0)\}|A\rangle. \tag{3}$$

Note that $M(1.5)_V$ is proportional to the lepton momentum l^μ; to obtain terms of zeroth order in l, it is sufficient therefore to extract the terms of order l^{-1} from the integral in Eq. (2). Such terms can arise only from the Born term[7] in $V^{\lambda\nu}$, and therefore can be computed! One of us (A.S.) has shown explicitly that in the zero-lepton-momentum limit the entire contribution of $M(1.5)_V$, $M(1.3)$, $M(1.4)$, and $M(1.6)$ to the decay rate reduces to the term of order $Z\alpha$ in the Coulomb function plus the usual radiative corrections[8] of order α. The Coulombic correction, however, is already included in the calculation of f; theorem I is therefore established.

Multiphoton effects.—Z-dependent renormalization effects do rear their ugly head in fourth order. How does one avoid a catastrophic enhancement of these effects by coherent emission and reabsorption of multitudes of virtual quanta? (Very naively each quantum will multiply the amplitude by a factor $Z^2\alpha$.)

We give a resolution of this problem for a particular type of radiative correction, one in which (a) no photons hit the electron line and (b) there are no vacuum polarization effects. (In the interest of brevity we procede as if mass counter terms were not necessary.) It will be clear immediately that all possible radiative corrections lend themselves to a similar treatment.[9]

The purely hadronic radiative corrections are contained in the matrix element

$$M_\mu = \langle B \text{ out}|\, T\left\{V_\mu{}^{(+)}(0)\exp\{-i\int[H^v(x)+H^s(x)]d^4x\}\right\}|A \text{ in}\rangle$$
$$=\sum_{n,m}\frac{(-i)^{n+m}}{n!\,m!}\int\langle B \text{ out}|\,T\{V_\mu{}^{(+)}(0)H^v(x_1)H^v(x_2)\cdots H^v(x_m)H^s(y_1)H^s(y_2)\cdots H^s(y_n)\}|A \text{ in}\rangle$$
$$\times d^4x_1d^4x_2\cdots d^4x_md^4y_1d^4y_2\cdots d^4y_n. \tag{4}$$

Here $H^i(x)\equiv J_\mu{}^i(x)A^\mu(x)$ ($i\equiv s$ or v), A^μ being the electromagnetic field. The states A and B are chosen to be eigenstates of, and all current operators are taken to be Heisenberg with respect to, the free plus the strong Hamiltonian.

The term explicitly exhibited in Eq. (4) is potentially of order $(Z\alpha)^{(m+n)/2}Z^{(n-m)/2}$ [$m+n$ is always even!]; it is therefore the terms with $m<n$ that require careful consideration. Our procedure for these terms is as follows:

(i) For any fixed $m<n$, contract the field operator A_μ which occurs in any H^v with every A_ν in the T product; i.e., make all possible contractions of pairs[10] of A's, omitting those pairs however in which each A emerged from an H^s.

(ii) Perform the summation over n, so that the H^s formally resum into an exponential inside the T product.

(iii) Perform a canonical transformation[11] so that the field operators in a new representation are Heisenberg operators with respect to free plus strong Hamiltonians plus $\int H^s(x)d^3x$.

This procedure gives

$$\delta M_\mu = \frac{(-i)^2}{2!}\int\langle\tilde{B} \text{ out}|\,T\{\tilde{V}_\mu{}^{(+)}\tilde{J}^v(x_1)D(x_1-x_2)\tilde{J}^v(x_2)\}|\tilde{A} \text{ in}\rangle d^4x_1d^4x_2+(-i)^2$$
$$\times\int\langle\tilde{B} \text{ out}|\,T\{\tilde{V}_\mu{}^{(+)}\tilde{J}^v(x_1)D(x_1-y_1)\tilde{J}^s(y_1)\}|\tilde{A} \text{ in}\rangle d^4x_1d^4y_1+\cdots. \tag{5}$$

In Eq. (5) $D_{\mu\nu}$ is the photon propagator and the tilde implies that the operators and state vectors are in the new representation. Note that passage to this representation does not affect the isotopic transformation properties of any operator.

It is clear that in Eq. (5) there can be no terms in which the power of Z exceeds the power of α. All such terms in Eq. (4) have been absorbed into the definition of the field operators and state vectors, effectively becoming part of the strong interactions!

Remarks.—(i) In our discussion of one-photon effects, we did not consider the contribution from Fig. (1.5) in which the axial current acts at the weak vertex. We have estimated this term in the simple-minded independent-particle model, and have convinced ourselves that because of the magnetic nature of these photons no coherent effects in the charge of order $Z\alpha$ can arise from it. The same conclusion was arrived at by Durand et al.[7] [see Eq. (123)]. (ii) We close with a word of caution: For large-Z nuclei, it is not very consistent to compute elaborate nuclear effects of a percent or so in the Fermi function, while neglecting unknown radiative corrections of order $Z^2\alpha^2$.

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[1]R. Blin-Stoyle, Phys. Letters 29B, 12 (1969). To the eight transitions listed in this paper, we have added $\pi^+\to\pi^0$ [$ft\cong 3100$].

[2]R. P. Feynman and M. Gell-Mann, Phys. Rev. 109, 193 (1958); S. Gerstein and J. Zeldovich, Zh. Eksperim. i Teor. Fiz. 29, 698 (1955) [translation: Soviet Phys.–JETP 2, 576 (1956)].

[3]In this theorem, a quantity $f(l_\mu)$ is understood to be of zeroth order in the lepton momenta if

$$\lim_{\zeta\to 0} f(\zeta l_\mu)\neq 0.$$

The theorem means that if we write the ft value in the form

$$ft=\frac{2\pi^3\ln 2}{G_V{}^2\cos^2\theta\,|M_V|^2(1+\delta)m_e{}^5},$$

then subject to (a), (b), and (c), δ is independent of Z. Note that this theorem is very closely related to the theorem of E. S. Abers, D. A. Dicus, R. E. Norton, and H. R. Quinn, Phys. Rev. 167, 1461 (1968).

[4]That the renormalization effects of mass shifts on a conserved current are of second order was first shown by R. E. Behrends and A. Sirlin, Phys. Rev. Letters 4, 186 (1960). The theorem was discussed independently by several authors in somewhat different contexts: M. V. Terent'ev, Zh. Eksperim. i Teor. Fiz. 44, 1320 (1963) [translation: Soviet Phys.—JETP 17, 890 (1963)]; M. Ademollo and R. Gatto, Phys. Rev. Letters 13, 264 (1964); C. Bouchiat and Ph. Meyer, Nuovo Cimento 34, 1122 (1964).

[5]See, e.g., M. A. B. Bég, Summer Institute of the Niels Bohr Institute, Copenhagen, Denmark, Lectures, 1967 (unpublished).

[6]See, e.g., G. Preparata and W. Weisberger, Phys. Rev. 175, 1965 (1967).

[7]The essential reason is that the Born contribution is sufficiently singular as $k \to 0$ so that the lepton momenta may not be neglected inside the integral. This particular argument has been discussed independently by several authors: A. Sirlin, Phys. Rev. 164, 1767 (1967); Abers, Dicus, Norton, and Quinn, Ref. 3; and J. D. Bjorken, Lectures on the Selected Topics in Particle Physics, International School of Physics "Enrico Fermi," Course XLI, edited by J. Steinberger (Academic Press, Inc., New York, 1967). These arguments do not exclude the possible and very probable existence of terms of order $Z\alpha l/M$, $Z\alpha l \ln(l)/M$, $Z\alpha(p'-p)/M$, etc., where M is some hadronic mass. Such terms are of higher order in the lepton momenta and, in the language of the second paper cited above, can be regarded as being of higher order in α. These terms have been exhibited in a computation of L. Durand et al., Phys. Rev. 130, 1188 (1963).

[8]The detailed argument will be published elsewhere by A. S.

[9]I.e., equations analogous to Eq. (5) below can be derived for all radiative corrections. Full details will be published elsewhere.

[10]Note that pairwise contraction of the A's in Eq. (4) does not yield the full Feynman amplitude in order e^{m+n}. It does, however, yield the part of the amplitude in which we are interested.

[11]Our procedure here is a straightforward extension of procedures described in standard texts on field theory. See, e.g., S. S. Schweber, An Introduction to Relativistic Quantum Field Theory (Harper and Row Publishers, Inc., New York, 1964), p. 690. We will not enter here into the difficult problem of discussing whether these transformations exist in the mathematical sense. Note also that our discussion of theorem II assumed that the states $\tilde{A}$ and $\tilde{B}$ cannot decay by emission of isoscalar photons. This does not introduce any error in our work since all the relevant nuclear states have this property (save Al^{26} which is metastable).