rise to the necessary thermal gradient.

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UNIVERSAL EIGENMODE IN A STRONGLY SHEARED MAGNETIC FIELD*

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It is shown, contrary to previous work, that in the presence of large shear ($L_s/R_p < R_p/a_i$) an unstable universal eigenmode exists. The criterion for the stabilization of this mode for long wavelengths ($k_\perp a_i \lesssim 1$) is $L_s/R_p < (M/m)^{1/3}$ which is more restrictive than the usual criterion for stabilization of the transient (convective) modes ordinarily considered.

For the past several years, widespread interest has been generated from the assumption that the universal mode[1] in a magnetic field with sufficient shear[2] only exhibits a transient (convective) instability since unstable eigenfunctions have not been found.[2,3] Instead it was observed that, for the universal and other related modes, a wave packet amplifies as it propagates[4-10] and therefore a system was considered unstable if (1) the fields amplify to a level detrimental to confinement, or (2) nonlinear wave reflections occur that establish a standing wave.[11]

Contrary to previous work, we show in this note that a normal mode always occurs. From this mode we obtain a stability criterion that requires more shear to stabilize the universal instability than hitherto predicted, and thus the previous criteria are irrelevant.

We consider the collisionless limit here; however, similar arguments pertain to the collisional case. We analyze the slab model of Krall and Rosenbluth,[2] where the velocity distributions are Maxwellian, the magnetic field is given by $B = B_0[\hat{z} + (x/L_s)\hat{y}]$, and the density varies in the x direction. The perturbation is taken to be $\varphi(\vec{r}, t) = \varphi(x)\exp(-i\omega t + ik_y y)$, and the equation governing the potential φ can then be written as the following operator equation[2]:

$$\left[1 + \eta + \frac{\omega - \omega_i^*}{k_\parallel v_{\mathrm{th}i}} Z\left(\frac{\omega}{k_\parallel v_{\mathrm{th}i}}\right) I_0(\vec{b}) e^{-\vec{b}} + i\eta \frac{\omega - \omega_e^*}{|k_\parallel v_{\mathrm{th}e}|} \exp\left(-\frac{\omega^2}{2k_\parallel^2 v_{\mathrm{th}e}^2}\right)\right]\varphi = 0,$$

$$\vec{b} = k_y^2 a_i^2 + k_x^2 a_i^2 = b - a_i^2 \partial^2/\partial_x^2, \quad Z(\xi) = \frac{1}{(2\pi)^{1/2}} \int \frac{dx}{x - \xi} \exp\left(-\frac{x^2}{2}\right). \tag{1}$$

The subscripts i and e refer to ions and electrons, respectively, $\eta = T_i/T_e$ is the ion-electron temperature ratio, $\omega^* = -k_y v_{\mathrm{th}}^2/\omega_c R_p$, $k_\parallel = k_y x/L_s$ is the diamagnetic drift velocity, $R_p = |n(x)/n'(x)|$ is the scale length of the plasma and $n(x)$ the plasma density, ω_c is the gyrofrequency, v_{th} is the thermal velocity, and a is the gyroradius. In addition, we have made the following standard assumptions: $k_y\lambda_D, k_y a_e, \omega/\omega_c, \omega/k_\parallel v_{\mathrm{th}e} < 1$. Also we consider $L_s/R_p < 4\sqrt{2}R_p/a_i$ since it was in this limit that previous analyses assumed that the normal modes vanished, and in this range the spatial variation of R_p can be neglected.

To analyze Eq. (1) we need only make a simple expansion in x and d^2/dx^2 which is valid in the limit $k_x a_i, k_\parallel v_{\mathrm{th}i}/\omega < 1$, which will be justified a posteriori. We then obtain

$$\left[\frac{d^2}{dx^2} - \frac{1}{\bar{a}_i^2} Q(x)\right]\varphi(x) = 0, \tag{2}$$

where

$$Q(x)=Q_R(x)+iQ_I(x),\quad Q_R(x)=-1+\frac{1+\eta}{I_0e^{-b}}\frac{1}{1-\omega_i^*/\omega}\left[1-\left(\frac{k_yxv_{\text{th}i}}{L_s\omega}\right)^2\right],$$

$$Q_I(x)=\frac{\eta}{I_0e^{-b}}\frac{1}{1-\omega_i^*/\omega}\left(\frac{\pi}{2}\right)^{1/2}\left|\frac{\omega-\omega_e^*}{k_yxv_{\text{th}e}}\right|L_s,\quad \bar{a}_i^2=a_i^2\frac{d}{db}\ln I_0e^{-b}. \tag{3}$$

It is in the treatment of this differential equation that the previous investigations have gone wrong. Essentially they have demanded that a proper normal mode must be oscillatory in the vicinity of $x=0$ and evanescent for x large, and since this differential equation leads to the inverse behavior they argue that there are no normal modes. However, the above boundary conditions are not the most general ones; specifically, the proper conditions are waves with outgoing energy flux for large x. What is then found is an eigenmode of the system where in the interior a standing wave is set up and outside there are outgoing waves carrying off energy.[12] In the absence of an energy source such behavior would normally lead to a temporally damped mode. However, in the present analysis this damping mechanism competes with the instability mechanism (inverse electron Landau damping) and a new, correct stability criterion is obtained.

It should be emphasized that the approximations governing Eq. (2) need only apply internally to the region of wave reflection (of the order of the turning point) since outside, the solution to Eq. (2) connects on smoothly to the eikonal solution of Eq. (1) $[\varphi(x)\sim\exp i\int^x k_x dx]$. From Eq. (1) it is readily ascertained that for large x ($k_\parallel v_{\text{th}i}/\omega \gtrsim 1$) the outgoing wave, which was primarily oscillatory for smaller values of x, spatially damps because of the onset of ion Landau damping.

To analyze the above differential equation, we temporarily ignore $Q_I(x)$ [since it is small compared with the individual terms of $Q_R(x)$] and obtain as a solution (H_n is the Hermite polynomial)

$$\varphi=H_n((i\sigma)^{1/2}x)\exp(-\tfrac{1}{2}i\sigma x^2),\quad n=0,1,2,\cdots,\quad \sigma=\frac{1}{L_s\bar{a}_i}\left(\frac{1+\eta}{I_0e^{-b}}\frac{\omega}{\omega-\omega_i^*}\right)^{1/2}\frac{k_yv_{\text{th}i}}{\omega}. \tag{4}$$

Note the negative sign in the exponential; this is because the wave is backward, i.e., $(k_x/\omega)d\omega/dk_x<0$, and as previously stated, the proper boundary condition is outgoing waves at large x. Now the dispersion relation in this limit is of the form

$$-1+\frac{1+\eta}{I_0e^{-b}}\frac{\omega}{\omega-\omega_i^*}=-i\bar{a}_i\left(\frac{1+\eta}{I_0e^{-b}}\frac{\omega}{\omega-\omega_i^*}\right)^{1/2}\frac{k_yv_{\text{th}i}}{\omega L_s}(2n+1). \tag{5}$$

Since the right-hand side of Eq. (5) is small, we have with $\omega=\omega_0+\delta\omega$

$$\omega_0=-\frac{\omega_i^*I_0e^{-b}}{1+\eta-I_0e^{-b}},\quad \delta\omega=-i\frac{\bar{a}_i}{L_s}k_yv_{\text{th}i}\frac{1+\eta}{1+\eta-I_0e^{-b}}(2n+1)\quad\text{(damping)}. \tag{6}$$

To obtain the stability criterion we must compare Eq. (6) with the growth obtained from inverse electron Landau damping. We estimate the latter contribution by evaluating $Q_I(x)$ at the turning point which, in the present limit, is determined from the equation $Q_R(x_T)=0$ or

$$\left(\frac{k_yx_Tv_{\text{th}i}}{L_s\omega_0}\right)^2=-i\frac{\bar{a}_i}{L_s}\frac{k_yv_{\text{th}i}}{\omega_0}(2n+1)=-i\frac{R_p}{L_s}\frac{\bar{a}_i}{a_i}\frac{1+\eta-I_0e^{-b}}{I_0e^{-b}}(2n+1). \tag{7}$$

(Note that x_T is on the $-45°$ line in the complex x plane.) Finally, with the addition of $Q_I(x_T)$ we obtain the dispersion relation

$$-1+\frac{1+\eta}{I_0e^{-b}}\frac{\omega}{\omega-\omega_i^*}+i\left(\frac{\pi}{2}\right)^{1/2}\frac{\eta}{1+\eta}\left(\frac{1-\omega_e^*}{\omega_0}\right)\left|\frac{\omega_0L_s}{k_yx_Tv_{\text{th}e}}\right|=-i\frac{\bar{a}_i}{L_s}\frac{k_yv_{\text{th}i}}{\omega_0}(2n+1), \tag{5a}$$

from which we obtain as our stability criterion for the normal mode

$$\frac{L_s}{R_p}<\left(\frac{M}{m\eta\pi}\right)^{1/3}\frac{1+\eta-I_0e^{-b}}{I_0e^{-b}}\left(\frac{I_0e^{-b}}{1-I_0e^{-b}}\right)^{2/3}-\frac{d}{db}\ln I_0^{-b}(2n+1) \tag{8}$$

with $n=0$ the most serious mode.

Before proceeding, we must check to see whether our assumptions are correct; namely, is

$$(k_x\bar{a}_i)^2 \approx \left(\frac{k_\parallel v_{\mathrm{th}i}}{\omega}\right)^2 \equiv \left(\frac{k_y x_T v_{\mathrm{th}i}}{L_s\omega_0}\right)^2 < 1, \tag{9}$$

which is the condition that ion Landau damping be negligible? Inserting Eq. (8) into Eq. (7) we obtain

$$\left(\frac{k_y x_T v_{\mathrm{th}i}}{L_s\omega_0}\right)^2 = \left(\frac{\pi m\eta}{M}\right)^{1/3}\left(\frac{1-I_0e^{-b}}{I_0e^{-b}}\right)^{2/3},$$

which leads to the trivial constraint $b < M/m\eta$. Another negligible constraint demanding $k_\parallel v_{\mathrm{th}e}/\omega > 1$ yields $b > m\eta/M$.

From Eq. (8) we see that the condition for stabilization of the larger wavelengths ($b \gtrsim 1$) is of order

$$L_s/R_p < (M/m)^{1/3}, \tag{10}$$

which is more restrictive than previously derived criteria. Moreover, we see that effectively there is no stabilization for very large b ($1 \ll b < M/m\eta$).

In conclusion we have shown that: (1) Large shear does not eliminate normal mode solutions. (2) This solution predicts that shear is a relatively ineffective stabilization mechanism for the universal mode [$L_s/R_p < (M/m)^{1/3}$]. (3) The studies of convective modes (transient responses) lead to a more optimistic stability criteria [$L_s/R_p < (M/m)^{1/3} \times$ (No. of e-folding lengths)],[7] and hence is not the pertinent condition for stabilizing the universal mode.

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