one must use the more general result given in Eq. (5) of the present paper. We (DFS) wish to thank Dr. J. H. Weis for calling this problem to our attention.

[9]J. B. Bronzan, Massachusetts Institute of Technology Reports No. CTP-57 and No. CTP-61 (to be published); J. C. Taylor, Clarendon Laboratory Report No. 35/68 (to be published); P. K. Kuo and P. Suranyi, Phys. Rev. Letters 22, 1025 (1969).

[10]Here $\alpha_n^{\pm}$ and $\beta_n^{\pm}$ are the Regge trajectory and residue functions for the nth daughter trajectory which couples to states with parity $\pm(-1)^{j-v}$, $\tau_n=\pm1$ is the signature of the nth trajectory, and $v=0$ ($v=\frac{1}{2}$) for boson (fermion) trajectories. We have combined the particle spins using $3j$ symbols in a manner which is convenient for the study of the kinematic constraints at $s=0$ and pseudothresholds; S and S' are not the usual channel spins. The functions $e_{\lambda\mu}{}^j(z)$ are rotation coefficients of the second kind [M. Andrews and J. Gunson, J. Math. Phys. 5, 1391 (1964)],

$$e_{\lambda\mu}{}^j(z)=\tfrac{1}{2}e^{-i(\pi/2)(\lambda-\mu)}[\Gamma(j+\lambda+1)\Gamma(j+\mu+1)\Gamma(j-\lambda+1)\Gamma(j-\mu+1)]^{1/2}\left(\frac{z+1}{2}\right)^{(\lambda+\mu)/2}$$
$$\times\left(\frac{z-1}{2}\right)^{-j-1-(\lambda+\mu)/2}\frac{1}{\Gamma(2j+2)}\,{}_2F_1\left(j+\lambda+1,j+\mu+1;2j+2;\frac{2}{1-z}\right).$$

[11]The threshold constraints will be discussed separately (S. A. Klein, to be published).

[12]Cf., for example, E. J. Squires, Complex Angular Momentum and Particle Physics (W. A. Benjamin, Inc., New York, 1963).

[13]Klein, Ref. 11; L. Durand, III, P. M. Fishbane, and L. M. Simmons, Jr., to be published.

[14]The irreducible unitary representations of the Lorentz group are labeled by parameters j_0 and σ which give the values of the Casimir operators $\vec{J}^2-\vec{K}^2=j_0^2+\sigma^2-1$, $\vec{J}\cdot\vec{K}=-ij_0\sigma$, $j_0=0,\frac{1}{2},1,\cdots$, σ pure imaginary. The representation coefficients are given by matrix elements of the boost operators, $d_{ll'\mu}{}^{j_0\sigma}(\beta)=\langle j_0\sigma l\mu|e^{-i\beta K_3}|j_0\sigma l'\mu\rangle$. These functions are discussed, for example, by S. Ström, Arkiv Fysik 29, 467 (1965), and 33, 465 (1966); and by W. H. Greiman, thesis, Iowa State University, 1969 (unpublished), who use the notation $A_\mu{}^{ll'}(\beta,j_0,-i\sigma)$ for the same functions.

[15]Our interpretation of Eq. (2) differs profoundly in this respect from the interpretations given their (equivalent) results by Bronzan, Taylor, Kuo, and Suranyi, Ref. 9.

CROSSING-SYMMETRIC REGGE-BEHAVED AMPLITUDES WITH NONLINEAR TRAJECTORIES

Mahiko Suzuki
Department of Physics, University of Tokyo, Tokyo, Japan
(Received 12 May 1969; revised manuscript received 16 June 1969)

An explicit example of crossing-symmetric Regge-behaved amplitudes is constructed for nonlinear trajectories. The Regge behavior is proved for $\mathrm{Re}\,t\to+\infty$, except on the real axis, as well as for $\mathrm{Re}\,t\to-\infty$.

In this Letter we present an explicit example of crossing-symmetric, Regge-behaved amplitudes (so-called Veneziano representation[1]) for nonlinear trajectories. It is given in the form of an integral representation involving parameters associated wtih Regge trajectories. Both Regge asymptotic behavior and fixed-u behavior are proved along any direction on the complex plane except on the real axis.

Let us first construct an amplitude having poles at desirable locations with residues of the correct angular dependence, that is, an amplitude in which (i) poles should be located at $\alpha(s)$ and $\alpha(t)=n$, where n is a non-negative integer, and (ii) the residue at $\alpha(s)=n$ [or $\alpha(t)=n$] should be an nth polynomial of t (or s), barring ancestors. It is easily seen that the following integral representation meets these two conditions:

$$\mathcal{B}(-\alpha(s),-\alpha(t))=\int_0^1 dz\, z^{-\alpha(s)-1+\Delta\alpha(s)f(z)}\times(1-z)^{-\alpha(t)-1+\Delta\alpha(t)f(1-z)},\tag{1}$$

where α is a Regge trajectory which obeys a dispersion relation[2]

$$\alpha(s)=as+b+\frac{s}{\pi}\int ds'\frac{\mathrm{Im}\alpha(s')}{s'(s'-s)},\tag{2}$$

and $\Delta\alpha$ is the deviation of α from a linear trajectory, for which we assume in our representation

$$\lim_{|s|\to\infty}\Delta\alpha(s)/s=0.\tag{3}$$

The function $f(z)$ is to satisfy the following important properties:

$$f(0)=0 \text{ and } f(1)=1, \tag{4}$$

$$d^m f(z)/dz^m = 0 \text{ at } z=0 \text{ and } 1 \tag{5}$$

for an arbitrary positive integer m. A class of such functions can be constructed explicitly.[3] We shall choose here a function defined as

$$f(z)=\frac{1}{c}\int_0^z dx\,(-\ln x)^{\ln x}[-\ln(1-x)]^{\ln(1-x)}, \quad c=\int_0^1 dx\,(-\ln x)^{\ln x}[-\ln(1-x)]^{\ln(1-x)}. \tag{6}$$

It is straightforward to check the property (5) for the function $f(z)$ defined above. It is also evident that $f(z)$ is regular, when regarded as a function of a complex variable z, except at $z=0$, 1, and ∞.

Since there is no pinching singularity in (1), all singularities originate in the two end points. In the neighborhood of $z=0$, (1) is rewritten as

$$\int_0^1 dz\, z^{-\alpha(s)-1}(1-z)^{-\alpha(t)+\Delta\alpha(t)-1}\exp\{\Delta\alpha(s)f(z)\ln z+\Delta\alpha(t)[f(1-z)-1]\ln(1-z)\}. \tag{7}$$

Thanks to (5), the exponent $\Delta\alpha(s)f(z)\ln z+\Delta\alpha(t)[f(1-z)-1]\ln(1-z)$ approaches zero as $z\to 0$ faster than any finite power of z. The third factor of the integrand in (7) does not give rise to any singularity in a finite region of s and t. Expanding $(1-z)^{-\alpha(t)+\Delta\alpha(t)-1}$ around $z=0$, one finds poles at

$$\alpha(s)=n \text{ (non-negative integer)}, \tag{8}$$

with a residue

$$\frac{1}{n!}[\alpha(t)-\Delta\alpha(t)+1][\alpha(t)-\Delta\alpha(t)+2]\cdots[\alpha(t)-\Delta\alpha(t)+n] \tag{9}$$

which is certainly an nth polynomial of $\alpha(t)-\Delta\alpha(t)$ and therefore of t. The same argument follows with s and t interchanged, since (1) is symmetric under the interchange of s and t. We have thus proved that the representation (1) has poles at desirable locations with the correct angular dependence.

Our next task is to examine the high-energy behavior of (1). The conditions (i) and (ii) given at the beginning are not sufficient for producing the Regge asymptotic behavior. The behavior as $t\to-\infty$ is obtained in a straightforward way if $\Delta\alpha(t)$ satisfies (3); the dominant contribution comes from the integral over z in the region around the maximum of the exponent, namely, in the neighborhood of $z=0$. One immediately finds

$$\lim_{t\to-\infty}\mathcal{B}(-\alpha(s),-\alpha(t))=\int_0^1 dz\, z^{-\alpha(s)-1}\exp\{[\alpha(t)-\Delta\alpha(t)+1]z\}=\Gamma(-\alpha(s))[-\alpha(t)+\Delta\alpha(t)]^{\alpha(s)}. \tag{10}$$

On the other hand the asymptotic behavior as $\mathrm{Re}t\to+\infty$ is less trivial to establish. An integral representation like (1) does not define a function in the region where $\alpha(t)\geqslant 0$ until it is continued analytically. The infinity point is an accumulation point of poles, or an essential singularity; so one must carry out analytic continuation to establish the same Regge behavior in the limit $\mathrm{Re}t\to+\infty$. This problem did not come out explicitly in the Veneziano representation since it was overcome by use of the well-known analytic property of the gamma function.

To investigate the asymptotic behavior as $\mathrm{Re}t\to+\infty$, we rewrite (1) using the substitution $z=1-e^{-y}$ as

$$\int_0^\infty dy\,\exp\{[\alpha(t)-\Delta\alpha(t)f(e^{-y})]y+[-\alpha(s)+\Delta\alpha(s)f(1-e^{-y})-1]\ln(1-e^{-y})\}. \tag{11}$$

We regard (11) as a function of the complex variable y. The integration over y is along the real axis. For $\frac{1}{2}\pi<\arg t<\frac{3}{2}\pi$ the integral as it stands is convergent because of (3), (4), and (5), thus giving the asymptotic behavior of (10) again. For the other range of the angle $-\frac{1}{2}\pi<\arg t<\frac{1}{2}\pi$ we rotate the path of the integral over y from the real axis as is shown in Fig. 1. If we are allowed to rotate it up to the ray with $\arg y=\delta$, we are able to establish the Regge-asymptotic behavior in the region $\frac{1}{2}\pi-\delta<\arg t<\frac{3}{2}\pi-\delta$, for we can repeat the argument leading to (10). To complete the rotation, we must make sure that the contributions from the large and small arcs vanish as $R\to\infty$ and $r\to 0$, respectively, and that under the rotation the path does not pass over singularities marring the Regge behavior. The former is, in fact, realized since the integrand in (11) is es-

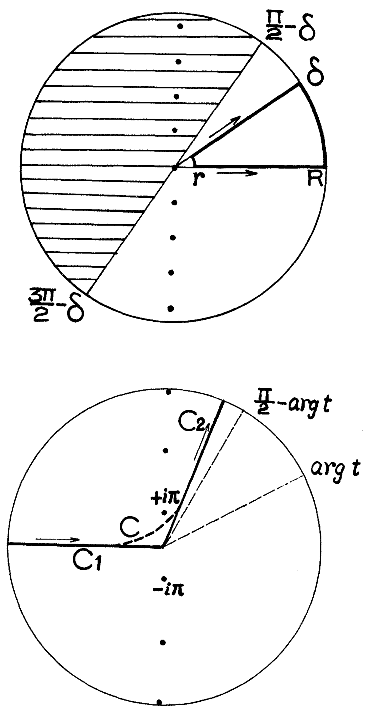

FIG. 1. (a) Rotation of the integral path. The hatched hemisphere is the angular range of the variable t where the Regge asymptotic behavior is proved. Singularities are located on the imaginary axis of the complex y plane. (b) Rotation and deformation of the integral paths on the w plane.

timated as

$$\int_{|y|=R} dy \to \int_0^\delta d\theta\, \mathrm{Re}^{i\theta} \exp\{[\alpha(t)-\Delta\alpha(t)f \times(\exp(-\mathrm{Re}^{i\theta}))]\,\mathrm{Re}^{i\theta}\},$$

where $|f(\exp(-\mathrm{Re}^{i\theta}))|$ behaves like $\exp(-R\cos\theta \times \ln R)$ on the large arc; so it goes like $\sim\int_0^\delta d\theta R \times\exp[|\alpha(t)|R\cos(\varphi+\theta)]$, where $\varphi=\arg t$, as far as $|\delta|<\frac{1}{2}\pi$. Therefore it vanishes as $R\to\infty$. Likewise, the contribution from the small arc is also shown to vanish as $r\to 0$ if $|\delta|<\frac{1}{2}\pi$ and $\alpha(s)<0$. As for singularities passed over by the path, one can easily see from the analytic property of $f(e^{-y})$ that there is no singularity in the first or fourth quadrant. Since $\ln(1-e^{-y})$ as well as the function f produces singularities on the imaginary axis, $|\delta|=\frac{1}{2}\pi$ is the boundary of the rotation of the path.

In this way we have proved the Regge asymptotic behavior along any direction on the first sheet of the complex t plane except on the real axis:

$$\lim_{|t|\to\infty} \mathfrak{B}(-\alpha(s),-\alpha(t))=\Gamma(-\alpha(s)) \times(-\alpha(t)+\Delta\alpha(t))^{\alpha(s)}. \tag{12}$$

More careful treatment will be needed to derive the Regge asymptotic behavior right along the real axis, just as in the case of the Veneziano representation. We shall not go into this program in the present paper. It is worth mentioning here that by looking at nonleading terms in the Regge asymptotic expansion we find no contribution characteristic of usual Regge cuts in our representation. On the other hand, Roskies has found terms like Regge-cut contributions in his model. The discrepancy originates obviously in that his model involves ancestors and exhibits the Regge behavior like $[-\alpha(t)]^{+\alpha(s)}$, instead of $[-\alpha(t)+\Delta\alpha(t)]^{+\alpha(s)}$ as appears in our case. Our present example, therefore, indicates that once one eliminates ancestors, one may have quite a different l-plane analyticity.

Finally, we make sure that (1) damps fast enough as $|t|\to\infty$ with u fixed. Because of the symmetry in s and t, it is sufficient to look at $\mathrm{Re}t\to+\infty$ only. In line with our having not yet established the Regge behavior along the positive real axis, we shall show it except along the real axis. For this purpose we rewrite (1) in terms of a new variable

$$w=\ln z/(1-z). \tag{13}$$

The integral over w then extends from $-\infty$ to $+\infty$. We split it into two parts:

$$\mathfrak{B}(-\alpha(s),-\alpha(t))=(\int_{-\infty}^0+\int_0^{+\infty})dw. \tag{14}$$

The first part is well defined as $\mathrm{Re}t\to+\infty$, while the second must be continued analytically. The singularities of the integrand are located at $w=\pm(2n+1)\pi i$ ($n=0,1,2,\cdots$) and ∞ on the w plane. The integral path of the second part is rotated as is shown in Fig. 1(b) until it makes the integral convergent as $\mathrm{Re}t\to+\infty$ (the contour C_2). The contribution from the large arc vanishes as before, and we are able to choose a new path C_2 without encountering the singularities on the imaginary axis, as far as $\mathrm{Re}t\to+\infty$, except along the real axis. Deforming further by a little bit the new contour consisting of C_1 and C_2 [see the contour C in Fig. 1(b)], we find the leading contribution comes from the region of the integral around either $w=\pm\pi i$, depending upon $\arg t$.

We thus obtain the asymptotic behavior as $\mathrm{Re}t\to+\infty$ with u fixed as

$$|\mathfrak{B}(-\alpha(s),-\alpha(t))|<e^{-a(\pi-\epsilon)|\mathrm{Im}t|}, \tag{15}$$

where ϵ is an arbitrarily small positive number and a is the slope of the linear part of the trajectory. This behavior is essentially the same as

Veneziano's, although we do not obtain an explicit form. We do not need any restriction stronger than (3) in deriving the correct asymptotic behavior with u fixed. A stronger restriction might be imposed if one requires the good behavior along the real axis.[4] It is necessary for the purpose of physical application to polish our representation so as to validate the asymptotic behavior along the real axis. But we believe that the present formula will be the first step towards constructing more realistic formulas with nonlinear trajectories.

The present representation reduces to the beta function in the limit of a linear trajectory $\Delta\alpha = 0$. It is straightforward to construct an analog to the Lovelace-Veneziano formula[5] out of our formula. Since we have introduced widths into resonances, we are in a position to impose elastic unitarity and to fix partly the ambiguity of adding satellites. But the present formula is still imperfect in that the degeneracy is not resolved completely among masses or total widths.

The author is grateful to H. Sugawara for helpful comments and to K. Kikkawa for useful correspondence.

[1]G. Veneziano, Nuovo Cimento 57A, 190 (1968).

[2]We have assumed, in accord with (3), a once-subtracted dispersion relation for $\alpha(s)$ with a linear dependence in front. The restriction will turn out to play an important role later in deriving the Regge behavior. In this connection see R. Z. Roskies, Phys. Rev. Letters 21, 1851 (1968).

[3]A class of functions satisfying (5) is sometimes called van der Corput's neutralizer. See, for example, E. T. Copson, Asymptotic Expansion (Cambridge University Press, London, England, 1965), p. 24. Why we use (6) instead of the more familiar form

$$\frac{1}{c}\int_0^z dx \exp\left(-\frac{1}{x}-\frac{1}{1-x}\right)$$

is to improve the behavior at the essential singularities $z=0$ and 1. This milder behavior at $z=0$ and 1 is indispensable in deriving the Regge behavior as Re$t \to +\infty$.

[4]It is also likely that a modification may be necessary for the neutralizer $f(z)$.

[5]C. Lovelace, Phys. Letters 28B, 265 (1968).

ERRATA

BROKEN-DUALITY MODEL FOR THE REACTION $pp \to \pi^+ d$. V. Barger and C. Michael [Phys. Rev. Letters 22, 1330 (1969)].

The data at 21.1 GeV/c attributed to Allaby et al. (Ref. 2) were taken from a preliminary version of the paper. The residue parameters of Eq. (6) obtained from a fit to the data as given in the published version are $a(N_\alpha) = -0.94$, $a(N_\gamma) = 8.7$, $b(N_\alpha) = -1.40$, and $b(N_\gamma) = 0.23$.

QUASIELASTIC RAYLEIGH SCATTERING IN NEMATIC LIQUID CRYSTALS. Orsay Liquid Crystal Group [Phys. Rev. Letters 22, 1361 (1969)].

The three textual mentions of Ref. 1 on page 1363 (lines 17 and 24 of the first column, and line 2 of the second column) are erroneous; they should read "Ref. 3."