PHOTOPRODUCTION OF MUON PAIRS: $\rho^0 \to \mu^+ + \mu^-$, (e, μ) UNIVERSALITY, AND (ρ, ω) PHASE*

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Data on photoproduction of muon pairs are reported and compared with quantum electrodynamics. A new average value of $B_\rho = (\rho^0 \to \mu^+ + \mu^-)/(\rho^0 \to \text{all}) = (8.2 \pm 1.6) \times 10^{-5}$ is obtained. Comparison with B_ρ for electron pairs provides a valid experimental confirmation, to "distances" of 1.6×10^{-14} cm, of (e, μ) universality for vertices involving timelike photons. We interpret the paradoxical agreement of values of B_ρ for photoproduction experiments with those for colliding beams in terms of the ρ, ω phase.

We report here on the results of an experiment on photoproduction from carbon of wide-angle muons (WAM), performed at Cambridge Electron Accelerator using a 6-GeV bremsstrahlung beam. Muons were identified by their ability to penetrate iron. This experiment, herein called WAM III, was done at higher γ energy and muon-pair invariant mass than our earlier work.[1] Previously a portion of the data of WAM III was used to set an upper limit on the branching ratio of the φ meson into muon pairs.[2,3] In the present note we report on the analysis of the remaining useful WAM III data. We restrict ourselves to events for which $|q_N| < 0.6\,\text{F}^{-1}$, where q_N is the nuclear recoil momentum, in order to reduce the uncertainty in the results due to inelastic events. We discuss here the bearing of the results upon quantum electrodynamics (QED), the branching ratio of the ρ meson into muons $[B_\rho \equiv (\rho^0 \to \mu^+ + \mu^-)/(\rho^0 \to \text{all})]$, and e, μ universality. We also consider the apparent paradoxical agreement in the world-wide values of B_ρ for photoproduction experiments with those for colliding-beam experiments and a possible resolution of the paradox.

The experimental equipment has been described previously.[1,4] The present experiment differs only in that individual muons were observed in the energy interval 2237 MeV $< E_\mu <$ 2785 MeV (in five equal subintervals), in the polar angular interval $5.16° < \theta_\mu < 11.84°$ (in nine equal subintervals), and in an azimuthal interval of 42° centered about 180° (in seven equal subintervals). The resulting invariant mass of the observed muon pairs covered the region 470 MeV $< M <$ 1070 MeV. Figure 1 shows the geometry of the iron blocks and detectors. Corrections applied to the

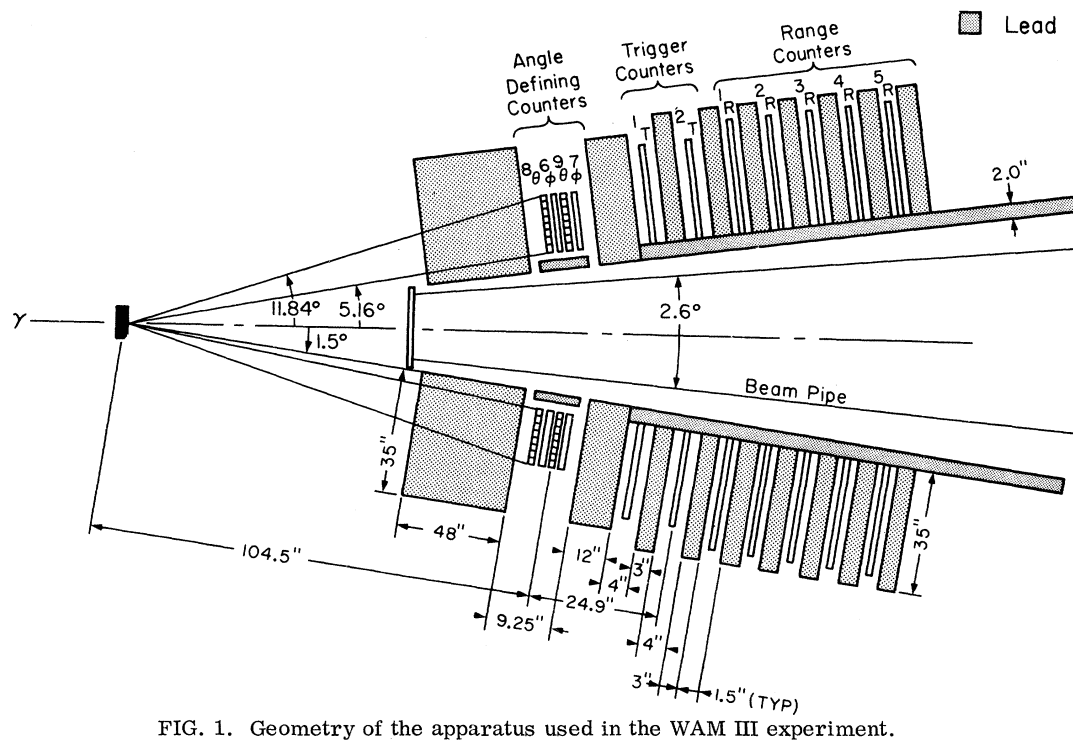

FIG. 1. Geometry of the apparatus used in the WAM III experiment.

data have been described previously,[1] and include corrections for target-out rates, chance coincidence events between the two arms, geometrical and electronic efficiencies, deadtime losses, and background due to π pairs and their decay products.

The theoretical muon yields are dominated by Bethe-Heitler diagrams and ρ^0 decay into muons.[1] The ρ^0 decay is also mainly responsible for our experimental background due to pion pairs. Both the Bethe-Heitler theory and the ρ theory have been previously discussed.[1] The $\varphi \to \mu^+ + \mu^-$ contribution has been neglected here because of its negligible effect on the results.[2] The theoretical yields have the form $Y_{\mathrm{theor}} = Y_{\mathrm{BH}} + B_\rho Y_\rho$. Y_{BH} is the yield due to the elastic and inelastic Bethe-Heitler processes, B_ρ is the ρ^0 branching ratio for muon pairs compared with pion pairs, and Y_ρ is the ρ^0-to-two-muon yield for a branching ratio of 1. Y_ρ and Y_{BH} do not interfere.

Work by Kroll[5] shows that deviations from theory must vary at least as fast as q_μ^4, where q_μ is the four-momentum transferred to the virtual muon. For muons symmetric about the γ beam, $M^4 \approx 4q_\mu^4$, where M is the invariant mass of the muon pair. For asymmetric muons this relationship is not valid. However, the differences are not significant and for the purposes of using a standard notation within the entire class of lepton-pair experiments[1-4,6] we will interpret our results in terms of a QED breakdown proportional to M^4. (For purposes of comparison with earlier work we also compare with a breakdown proportional to M^2.) We include a term $\beta M^4 Y_{\mathrm{BH}}$ in the theory where β is a parameter which, if different from zero, indicates a breakdown in QED. Also in order to allow for a difference in normalization between experiment and theory, we include a constant A. This puts the theory in the form $Y_{\mathrm{theor}} = A(\{1+\beta M^4\} Y_{\mathrm{BH}} + B_\rho Y_\rho)$. Finally, it is convenient for comparison with most other work to express the theory as a ratio, R_{theor}, to the Bethe-Heitler theoretical yields. Thus

$$R_{\mathrm{theor}} = A(1+\beta M^4 + B_\rho\{Y_\rho/Y_{\mathrm{BH}}\}). \qquad (1)$$

In general, the values of the constants obtained from a best fit of this expression to the data will vary with the power of M assumed for the breakdown model. On a graph of R vs M, A is the intercept on the R axis (i.e., for $M=0$) of the best fit. Unlike an ordinary normalization constant the resulting value of A is strongly model dependent.

We compare the theory of Eq. (1) with the experimental yields, where the latter are also divided by $Y_{\mathrm{BH,theor}}$. The adjustable parameters A, β, and B_ρ are determined by a best fit of experiment and theory.

The best fits of the WAM III data with Eq. (1) for M^2 and M^4 slopes are given in Table I. The results of WAM II[1] are included for comparison. The experimental results are shown in Fig. 2.

QED.—We consider first the consistency of WAM III and WAM II. We note that the values of the slopes, β, in WAM III and II are in good agreement. The value of β, if nonzero, indicates either a disagreement with QED or an unsuspected source of systematic error. In both experiments we see that β is nonzero, and negative, by about 2 error measures.[7] It thus appears that the WAM III data confirm the presence of a disagreement with QED as indicated in WAM II. However, there is some evidence that we are dealing not with a real slope, but with a systematic error. When, for given values of the mass, the absolute normalization of WAM III is compared with that for WAM II, the WAM III points

Table I. Best fits of data by Eq. (1).[a]

Experiment	Slope variable	A	β	$10^5 B_\rho$	χ^2/N
WAM III	M^2	1.49 ±0.25	−0.72 ±0.29 BeV^{-2}	7.4 ± 2.0	5.9/13
WAM II[b]	M^2	1.40 ±0.23	−0.75 ±0.37 BeV^{-2}	$4.4^{+2.1}_{-0.9}$	
WAM III	M^4	1.25 ±0.15	−0.80 ±0.43 BeV^{-4}	8.5 ± 2.4	4.8/13
WAM II[c]	M^4	1.27 ±0.16	−1.6 ±0.7 BeV^{-4}	$7.3^{+3.0}_{-1.5}$	

[a]Errors in A, β, and B_ρ include statistical errors and uncertainties due to the carbon form factor, muon energy, inelastic contributions, and experimental corrections enumerated in the text. A normalization error of 8.6%, added linearly rather than quadratically, is also included.

[b]In Ref. 1, the error in β for WAM II is given as ± 0.33 GeV^{-2}. This quoted error has been increased here as a result of studies of the energy-slope relationship discussed in the text.

[c]The errors of the M^4 fit to WAM II data are approximate. They have been estimated here rather than calculated since no M^4 fit was included in the analysis of Ref. 1.

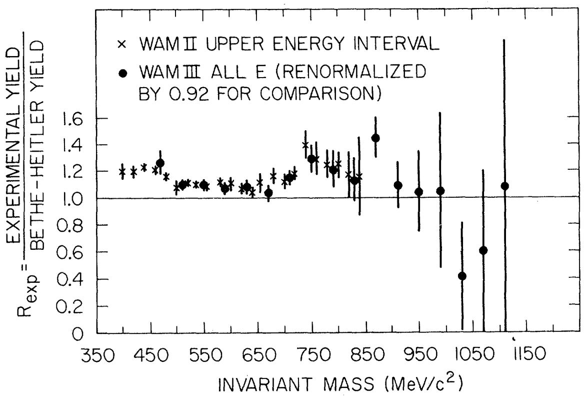

FIG. 2. Results of the WAM II and III experiments. Experimental yields have been divided by elastic plus inelastic theoretical yields from Bethe-Heitler processes, hence the yield from $\rho \rightarrow \mu^+ + \mu^-$ shows up as an excess. Corrections to the yields are enumerated in the text.

are found to be 7.5% higher. The absolute normalization errors, which apply to such a point-by-point comparison between these two experiments, are sizable. However, most of the error is systematic and the two experiments share most of the systematic errors in common. They should, in fact, be nearly identical except for the statistical errors. Including only the statistical errors, however, they differ by $(7.5 \pm 2.5)\%$. If the slope β were due to a hidden systematic error then when the spectrometers are reset to cover a higher mass interval, as in WAM III, the systematic error in β results in a higher value of the absolute normalization for WAM III points relative to WAM II points for those masses common to the two experiments. The high point-by-point normalization of WAM III with respect to WAM II may then be simply explained by assuming a hidden systematic error in β about twice as large as our best estimate of the error. It should be noted that if we are dealing with a systematic error, it is mainly in β, not in the average experimental normalization. The average absolute experimental normalization of the mass points below the ρ-meson region is within about one error measure of the theoretical value of 1.0.

We have searched for the source of the slope, assuming it to be a systematic error. Previously inelastic corrections were ruled out.[1] The present experiment has less than $\frac{1}{4}$ of the yield from π pairs and their decay products than was present in WAM II.[1] Since WAM II and III agree in slope, the π-pair background is ruled out as a significant source of this slope. We have studied the errors introduced by various range-straggling approximations and found that these leave β virtually unaffected. We have considered the reaction $\gamma + \text{C} \rightarrow \varphi + \text{C}$; $\varphi \rightarrow K^+ + K^-$; $K \rightarrow$ decay. This is a particularly attractive possibility since it would provide a slope, and would also provide an excess yield at low M when the K's and their decay products are interpreted as μ's. However, the results of a Monte Carlo calculation show that the yields due to this reaction are only about 1% in our lowest mass bins.

Finally, an investigation was made of the effects of an uncertainty in the assumed range-energy relationship, or an error in the measured range. Calculations were performed using two theoretical energy values differing by $\pm 1.3\%$. These show a slope introduced by a range-energy error. We had assumed an error of approximately 1% in the thickness of the iron absorbers in our earlier work. However, in order to estimate the effect of this uncertainty, we used the functional form of the Bethe-Heitler cross section for muons symmetric about the γ beam, for which an error in range causes an error in normalization, but not in slope. As a result of the full calculation performed in our more recent work, including the dominant asymmetric pairs to estimate the error in β, we find that an error of 1.3% in range contributes errors in slope of ± 0.33 GeV^{-4} and ± 0.16 GeV^{-2} to the M^4 and M^2 fits, respectively. The values of $\Delta\beta$ quoted in Table I include these errors, compounded quadratically. As a result of this, the quoted errors reported in Table I for WAM II are slightly higher than those appearing in the literature.[1] While the calculation described was performed for the WAM II geometry, it is also very nearly valid for the WAM III geometry. We also expect this effect to be present in the experiment of Hayes et al.[6] Its inclusion would significantly increase their quoted error of $\Delta\beta = \pm 0.05$ GeV^{-4}, and would also increase their normalization error.

Of the errors previously considered and corrected for, only the multiple Coulomb scattering corrections still remain suspect. These are being checked in experiments and calculations presently underway. Entirely different analysis programs are being written in order to check for undiscovered programming errors.

In summary, our present conclusion is that the results of WAM III are in agreement with those of WAM II, and there is good though not conclusive evidence that the slopes of WAM II and III are due to hidden systematic error. The errors of WAM II and III have had to be increased due to a previously unsuspected coupling of range and slope errors. The data of WAM III improve our probe of QED for an M^4 breakdown by about a factor of 2.

B_ρ, and e, μ universality.–Next we consider the values obtained for B_ρ in WAM II and III. These two values (Table I) differ by approximately one error measure or less for a given breakdown model and are therefore consistent. The best average value of B_ρ we obtain from the two experiments is

$$B_\rho = (8.2 \pm 1.6)\times 10^{-5} = (B_\rho)_{\text{ppt}}. \quad (2)$$

This value of B_ρ, and the value of γ_ρ to be obtained from it, must be considered in the light of remarks on ρ, ω interference which will be made later in this note. For this reason we use a subscript to indicate that these values result from the total μ pair yield from ρ's in a photoproduction experiment. We use in Eq. (2) the results for $B_4 \equiv B_\rho$ obtained from the M^4 fits to WAM II and III, whereas in earlier results[1,3] $B_2 \equiv B_\rho$ obtained from the M^2 fits was reported. The use of B_4 is indicated as the proper procedure by the fact that (a) if the value of β is correct, and a limit to QED is being observed, the minimal power of the mass involved[5] should be M^4. (b) If, alternatively, β is due to a systematic error, the higher the value used for the power of the mass, the less dependent A is on the errors in β. Thus whether the observed slope is due to physics or to a systematic error a better value of B_ρ will emerge from B_4 than from B_2. The value of B_ρ in Eq. (2) corresponds to a ρ, γ coupling constant[8]

$$(\gamma_\rho^2/4\pi)_{\text{ppt}} = 0.39_{-0.06}^{+0.09}. \quad (3)$$

The concept of e, μ universality can be checked in the timelike domain by comparing $B_\rho \to \mu^+ + \mu^- = B_{\rho\mu}$ with $B_\rho \to e^+ + e^- = B_{\rho e}$.[3] For $B_{\rho e}$ we use the value from Deutsches Elektronen Synchrotron (DESY)[9] rather than those of the colliding beams, in order to eliminate the unknown effects of ρ, ω interference (which, as mentioned above, obscure the direct interpretation of B_ρ and γ_ρ) and perform a valid test of e, μ universality. The DESY value is $B_{\rho e} = (6.5 \pm 1.4)\times 10^{-5}$. Hence $B_{\rho\mu}/B_{\rho e} = 1.26 \pm 0.37$. If we assume a form factor f at the (lepton, lepton, γ) vertex, and set $f \approx 1/(1+q^2/\Lambda^2)$ where Λ is the timelike breakdown momentum for the vertex, then

$$\frac{B_{\rho\mu}}{B_{\rho e}} \approx \left(\frac{1}{1+q^2/D_t^2}\right)^2 \approx 1.26 \pm 0.37, \quad (4)$$

where $D_t^{-2} = \Lambda_\mu^{-2} - \Lambda_e^{-2}$. From Eq. (4) we find $D^2 > 10\ \text{GeV}^2/c^2$ [1 standard deviation (s.d.)] or

$$D^2 > 1.5\ \text{GeV}^2/c^2 \text{ (2 s.d. limit)}. \quad (5)$$

Equation (4) represents the only interpretable experimental data known to us which probes (e, μ) universality for vertices involving timelike photons. Experiments comparing μ, p scattering[10] with available accurate data on e, p scattering have probed the (lepton, lepton, γ) vertex for spacelike photons to $<0.9\times 10^{-14}$ cm (2 s.d.). The equivalent "distance" probed here is $<1.6 \times 10^{-14}$ cm (2 s.d.).

An apparent paradox (and its bearing upon tests of e, μ universality, γ_ρ, and ρ, ω mixing).–We note that the results for B_ρ as given in Eq. (2) agree to about 1 s.d. with the average of the colliding-beam results, $B_{\rho e} = (6.6 \pm 0.9)\times 10^{-5}$ from Augustin *et al.* and $B_{\rho e} = (5.0 \pm 1.0)\times 10^{-5}$ from Auslander *et al.*[11] They, of course, also agree with the DESY photoproduction results as indicated by Eq. (4). This situation may be viewed with some happiness as indicative of corroborative experiments, but we view the agreement between the colliding-beam results and the photoproduction results as somewhat of a paradox. Recent work[12] shows that if (a) one uses the measured values of B_ρ and B_ω,[10] (b) assumes that the ρ, ω relative phase is near zero as given by simple diffraction production models, (c) assumes that the ratio of *diffraction*-produced ρ's to *diffraction*-produced ω's does not differ markedly from 9:1,[13] and (d) assumes that the energy dependences of the ρ and ω diffraction production are similar, then one expects $B_{\rho e}$ measured from total $\rho \to e^+ + e^-$ yield in photoproduction experiments to be about 80% higher than $B_{\rho e}$ measured via colliding beams. If (e, μ) universality is valid one expects the same discrepency between the colliding-beam results and the photoproduction results for $B_{\rho\mu}$. This is due to the existence of large ρ, ω interference terms in the photoproduction experiments,[12] and their very small magnitude in colliding-beam experiments.[14] The assumptions (a)-(d) appear to us to be very reasonable. We therefore consider the agreement of the photoproduction and colliding-beam values of B_ρ to be surprising.

We wish to note here that there are three effects of the large expected interference terms. First, there is an effect on tests of e, μ universality. The present lack of experimental knowledge of the ρ, ω relative phase makes it *impossible to compare $B_{\rho\mu}$ with $B_{\rho e}$, in any two experiments where the interference terms may differ* significantly, without quoting an additional error of approximately ±80%. For most work this would make the total error in such a comparison near 100%. Thus in testing (e, μ) universality in the timelike domain, equivalent experiments must be compared, as is done in this note, and previously by Weinstein.[3] Comparisons of experiments with widely differing interference terms, such as was done by Wehmann et al.[15] and by Ting,[16] do not provide valid tests because of the present impossibility of calculating the ρ, ω interference term, and the resulting very large uncertainty. Such data can be used, in principle, to shed light upon the phase of the interference term, as is done below.

The second result of the ambiguity due to the interference terms, as it concerns the work presented here, is that $(B_\rho)_{\rm ppt}$ and $(\gamma_\rho)_{\rm ppt}$ presented in Eqs. (2) and (3) cannot be simply interpreted as the ρ branching and the ρ, γ coupling constant. The value of B_ρ, upon which the value of γ_ρ is based, is due not only to ρ decay into muon pairs but also to ω decay and to the ρ, ω interference terms. Until the phase of the ρ, ω interference terms is known, γ_ρ must be taken from colliding-beams results.[17] The electron-pair experiment, Ref. 9, similarly cannot be expected to lead to an interpretable value of B_ρ or γ_ρ. Therefore, until a proper interpretation of the ρ, ω interference terms is possible,[12] the values of $B_{\rho\mu}$ and $B_{\rho e}$ obtained in photoproduction experiments, and their resulting values of γ_ρ, should not be used in world average values.

Finally, we can compare the result for $(B_\rho)_{\rm ppt}$ given in Eq. (2) with that for B_ρ from colliding beams, and interpret the result as a measure of the ρ, ω phase. We take the average value of B_ρ from colliding beams to be $B_\rho = (5.9 \pm 0.7) \times 10^{-5}$. Using the results of Greenhut et al.,[12] the value of $(B_\rho)_{\rm ppt}$ found in this work then can be interpreted as yielding a ρ, ω phase of $100 \pm 35°$ or $305 \pm 35°$. These appear to be in disagreement with the simple diffraction model. (The small ρ, ω mixing observable in the colliding-beam experiments, due to mixing of the 2π and 3π decay modes,[14] tends only to increase the expected disagreement between photoproduction and colliding-beam experiments by ~4%.[12])

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[8]We use $\gamma^2/4\pi = \alpha^2 M_\rho / 12\Gamma_{\rho \to \mu\mu} = \alpha^2 M_\rho / 12 B \Gamma_{\rho, \rm tot}$ with $M_\rho = 765$ MeV and $\Gamma_{\rho, \rm tot} \approx 108$ MeV.

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FACTORIZATION PROPERTIES OF THE DUAL RESONANCE MODEL: A GENERAL TREATMENT OF LINEAR DEPENDENCES*

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In the dual resonance model, introducing a generalized gauge-transformation operator which leaves the vertex functions invariant, we present a general treatment of linear dependences among the vertex functions. We also prove that the two propagators recently discussed in connection with the twisting operator are equal up to our gauge transformation.

Soon after the proposal by Veneziano[1] of a four-point function satisfying duality as well as Regge behavior, a generalization to the N-point function was obtained.[2,3] More recently, it has been shown[4,5] that the dual resonance model proposed in Ref. 3 obeys factorization, in the sense that residues of the poles can be expressed in the form

$$\textstyle\sum_l f_l^*(\bar{q}, N_{\bar{q}}) f_l(p, N_p), \tag{1}$$

where p ($\bar{q}$) stands for the incoming (outgoing) monenta, N_p ($N_{\bar{q}}$) is the number of incoming (outgoing) lines, and the summation runs over a finite number of terms. If these functions f_l were linearly independent, then the number of terms appearing in the sum would specify the degree of degeneracies at the resonance. However, there exist linear relations among these functions, as already shown in Refs. 4 and 5. In this Letter we shall introduce a generalized gauge-transformation operator S and present a general treatment of the problem of linear dependences using this operator, which will enable us to obtain all the linear relations in a simple closed form.

It has proved very convenient to study the dual N-point function by employing the operator formalism recently proposed by Fubini, Gordon, and Veneziano.[6] According to these authors, one introduces a set of "harmonic-oscillator" operators $a_\mu^{(i)}$ obeying the usual commutation relations

$$[a_\mu^{(i)}, a_\nu^{\dagger(j)}] = \delta_{ij} g_{\mu\nu}, \tag{2}$$

and a Hilbert space generated by the repeated application of the operators $a_\mu^{\dagger(i)}$ on a "vacuum" vector $|\varphi\rangle$. Then the N-point function can be written as

$$A_{r+1,s+1} = \int dz\, z^{-\alpha(s)-1}(1-z)^{\alpha(0)-1}\langle \bar{q}| z^R |p\rangle, \tag{3}$$

where the momenta p and $\bar{q}$ are defined in Fig. 1(a), the operator $R = \sum_n n a^{\dagger(n)} a^{(n)}$, the vector $|p\rangle$ is given by

$$|p\rangle = \int dx\, \varphi(x,p) \exp\Big(\sum_{n=1}^{\infty} \sum_{l=0}^{r} \rho_l^{\,n} p_l \frac{a^{\dagger(n)}}{\sqrt{n}}\Big)|\varphi\rangle, \tag{4}$$

and an analogous expression is given for the vector $|\bar{q}\rangle$. In Eq. (4), $\varphi(x,p)$ is the integrand of the (r