preliminary study. A falloff in modulated emission intensity, expected when $\Delta f \tau_I \geqslant 1$, was observed to occur at $\Delta f \sim 2$ MHz on the 4200.7-Å line, in agreement with calculated lifetimes. This suggests that our technique may provide a simple way to measure lifetimes of optical or other transitions. In addition, since emission from ionic states should exhibit modulation in proportion to the ion density, identification of the electron and ion components of plasma oscillations could be possible. Finally, we wish to point out that the method may also be useful in overcoming "lifetime" limitations of other kinds, such as recombination radiation in solids.

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OPERATOR ALGEBRA AND THE DETERMINATION OF CRITICAL INDICES*

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The "reduction hypothesis" proposes that a product of nearby fluctuating local variables can be replaced by a linear combination of individual local variables. The linear combinations thereby produced are a kind of algebra of the reduction of products. A particular algebra is proposed for the two-dimensional Ising model. It is shown that a knowledge of which coefficients in the algebra are nonvanishing is sufficient to determine all critical indices.

The two-dimensional Ising model has a critical behavior which can be described in terms of a number of quasilocal fluctuating quantities such as the magnetization $\sigma_{\vec{r}}$; the energy density minus its critical value,[1] which we write as $\epsilon(\vec{r})$; a traceless and symmetric stress density[2] $t_{ij}(\vec{r})$; and a two-component fluctuating variable[3-5] $b_\pm(\vec{r})$ which has rotational properties like a spinor. The critical properties are defined by critical indices. According to the scaling idea[6,7] all critical indices would be determined if we knew a few fundamental "scaling indices" which describe how these fluctuating variables change under transformations of the scale of length.

The Onsager[8] solution of the two-dimensional Ising model provides us with a method of calculating these fundamental scaling indices. From this solution, we learn that $\sigma_{\vec{r}}$ scales as $r^{-1/8}$, $\epsilon(\vec{r})$ scales as r^{-1}, and $b_\pm$ scales[9] as $r^{-1/2}$. But, to obtain these results, we must rely upon the detailed calculations which grew from Onsager's original inspiration. These calculations have the drawback that they do not offer a very useful insight into the physics. Furthermore, they do not suggest any method of proceeding for other critical problems.

Only one fundamental index has been determined by a very direct method, the index for t_{ij}. Kawasaki has evaluated correlation functions involving $T_{ij} = \int d\vec{r}\, t_{ij}(\vec{r})$ by using the fact that this total stress tensor is conjugate to the strain. This evaluation implies that T_{ij} scales as r^0. Then t_{ij} will scale as r^{-2} in a two-dimensional system.

To determine all the other indices, we must try a new tack. In this paper, we make use of a "reduction hypothesis"[10] which suggests that a product of any two nearby fluctuating local quantities is expected to behave as a linear combination of all the other local variables.[11] In symbols,

$$O_\alpha(\vec{r}_1)O_\beta(\vec{r}_2)=\sum_\gamma A_{\alpha\beta,\,\gamma}(\vec{r})O_\gamma(\vec{R}),\qquad \vec{r}=\vec{r}_1-\vec{r}_2,\quad \vec{R}=\tfrac{1}{2}(\vec{r}_1+\vec{r}_2), \tag{1}$$

where the O's are the fluctuating variables and the A's are numbers which describe the structure of their algebra. It is the basic contention of this paper that a knowledge of the nature of the algebra, in particular which of the A's are nonvanishing, is sufficient to determine the critical indices.

To make this point in concrete fashion, I shall write down a set of algebraic relations of the form of Eq. (1) and then use them to determine all critical indices. These relations have been derived from a detailed calculation which will be published later—but their source is almost irrelevant to the main point, which is to show that reduction algebras can determine indices.

To write the algebra we use a basic set of variables $D_\gamma(\vec{r})$, where γ is a positive or negative integer or half integer. We allow $\vec{r}_1$ and $\vec{r}_2$ to be on the x axis with $x_1<x_2$. This algebra is basically the statement that as $\vec{r}_1$ and $\vec{r}_2$ approach one another at the critical point, $D_\alpha(\vec{r}_1)D_\beta(\vec{r}_2)$ is a constant times $D_\gamma(\vec{R}^*)$, where $\vec{R}^*$ is some point on the x axis in the neighborhood of $\vec{r}_1$ and $\vec{r}_2$ and

$$\gamma=\alpha+(-1)^{2\alpha}\beta. \tag{2}$$

More precisely, if $\gamma\neq 0$,

$$D_\alpha(\vec{r}_1)D_\beta(\vec{r}_2)=A_{\alpha\beta}(r)D_\gamma(\vec{R})+B_{\alpha\beta}(r)D_\gamma'(\vec{R})+\text{higher order in } r,\quad D_\gamma'(\vec{R})=\partial D_\gamma(\vec{R})/\partial X. \tag{3a}$$

The D' term represents the possibility of a displacement of $\vec{R}^*$ from $\vec{R}$. Since $D_0(\vec{R})$ is the unit operator, the case $\gamma=0$ is somewhat special. In this case, D_γ' vanishes and, instead,

$$D_\alpha(\vec{r}_1)D_\beta(\vec{r}_2)=A_{\alpha\beta}(r)D_0(\vec{R})+\tilde{B}_{\alpha\beta}(r)\epsilon(\vec{R})+C_{\alpha\beta}(r)t_{11}(\vec{R})+\text{higher order in } r. \tag{3b}$$

The detailed calculation also provides physical identifications of the D_α. The first three are

$$D_0=1,\quad D_{1/2}=\sigma_{\vec{r}},\quad D_{-1/2}=\mu_{\vec{r}}. \tag{4a}$$

Here $\mu_{\vec{r}}$ is a variable which has not been used up to now in most Ising-model studies. In the notation of Ref. 5, $\mu_{j,k}=\sqrt{2}\,b_{j,k}\sigma_{j,k-1}$. It turns out that $\mu_{\vec{r}}$ is the transform of $\sigma_{\vec{r}}$ under the Kramers-Wannier[12] transformation. This transformation interchanges the regions $T>T_c$ and $T<T_c$ and it represents an exact symmetry of the two-dimensional Ising model. Therefore $\mu_{\vec{r}}$ has for $T>T_c$ all the properties that $\sigma_{\vec{r}}$ has for $T<T_c$, and vice versa. Since the critical indices are the same above and below T_c, $\mu_{\vec{r}}$, and $\sigma_{\vec{r}}$ scale as the same power of r. They are also both scalars under rotation. Next $D_{\pm1}$ is identified by the exact calculations as the particular linear combination of the usual[3-5] Ising-model spinor variables which transform as spinor components with spin in the $\pm x$ direction. We write these special linear combinations of b_+ and b_- as $a_\pm$, so that

$$D_{+1}(\vec{r})=a_+(\vec{r});\quad D_{-1}(\vec{r})=a_-(\vec{r}). \tag{4b}$$

The detailed calculation also finds $D_{\pm3/2}$ as y derivatives of $D_{\pm1/2}$, in particular,

$$D_{\pm3/2}(\vec{r})=\pm c(\partial/\partial y)D_{\mp1/2}(\vec{r}), \tag{4c}$$

where c is a constant. We defer the discussion of $D_{\pm2}$ for a moment. For $|\gamma|>2$, D_γ is rather weakly fluctuating and we may take

$$D_\gamma=0 \text{ for } |\gamma|>2 \tag{4d}$$

for our present purposes.

The structure of the reduction algebra represented by Eqs. (2) and (3) can be partially understood from the spinor identification of $D_{\pm1}$. Imagine that we were in a three-dimensional system with spherical symmetry rather than a two-dimensional one. Then the spinor components $D_{\pm1}$ would have angular-momentum quantum number $J_x=\pm\frac{1}{2}$. The algebra would be reasonable if D_γ for integral γ had $J_x=\frac{1}{2}\gamma$. Then Eq. (2) would simply reflect angular-momentum addition properties for multiplication of operators on the x axis—at least for the case in which α and β were both integral. Of course, we are not dealing with a system with the full rotational symmetry of the three-dimensional case. Nonetheless, apparently enough rotational symmetry remains so that the combination rules of the D_γ closely resemble the algebra of operators with well-defined J_x.

Symmetry arguments provide additional information about the coefficients $A, B, \cdots$ in the algebra. Under $x\to -x$,

$$D_\gamma\to D_{\gamma'},\ \text{where } \gamma'=-(-1)^{2\gamma}\gamma, \tag{5a}$$

while the combined application of the Kramers-Wannier transform and also $y\to -y$ implies

$$D_\gamma\to D_{-\gamma}. \tag{5b}$$

These exact symmetries, for fixed r, result in the following relations among the coefficients:

$$A_{\alpha,\beta}=A_{-\alpha,-\beta}=A_{\beta',\alpha'},\quad B_{\alpha,\beta}=B_{-\alpha,-\beta}=-B_{\beta',\alpha'},\quad \tilde{B}_{\alpha,\beta}=-\tilde{B}_{-\alpha,-\beta}=\tilde{B}_{\beta',\alpha'},\quad C_{\alpha,\beta}=C_{-\alpha,-\beta}=C_{\beta',\alpha'}. \tag{6}$$

To find the r dependence of the coefficients, we consider the situation in which r is much greater than a lattice constant so that scaling concepts are applicable. Then, if D_γ scales as $r^{-\nu_\gamma}$ and ϵ scales[13] as $r^{-\nu_\epsilon}$, the scaling idea implies[14]

$$A_{\alpha,\beta}(r)\sim r^{\nu_\gamma-\nu_\alpha-\nu_\beta},\quad \tilde{B}_{\alpha,\beta}(r)\sim r^{\nu_\epsilon-\nu_\alpha-\nu_\beta},\quad B_{\alpha,\beta}(r)\sim r^{\nu_\alpha+1-\nu_\alpha-\nu_\beta},\quad C_{\alpha,\beta}(r)\sim r^{2-\nu_\alpha-\nu_\beta}, \tag{7}$$

Clearly, my physical understanding of the algebraic rules is still very imperfect. Nonetheless, it is possible to use these rules for the determination of all critical indices. The general scheme for this analysis is a three-step process: (i) Use the rules for operators which do not involve $\partial/\partial y$ to find the behavior of products on the x axis; (ii) from the known rotational properties of scalars, spinors, vectors, and tensors, generalize the result to the case $\vec{r}_1-\vec{r}_2$ arbitrary; and (iii) insist that the results found in part (ii) remain consistent with the multiplication rules for $D_{\pm 3/2}$ as they are defined in terms of y derivatives. As a first example of this process consider:

(a) $D_{\pm 2}$ and the index for $\epsilon(\vec{r})$.—When α and β are both $+\frac{1}{2}$, we can write Eq. (3b) as

$$\sigma_{\vec{r}_1}\sigma_{\vec{r}_2}=(w/r^{2\nu_{1/2}})[1+ur^{\nu_\epsilon}\epsilon(\vec{R})+vr^2t_{11}(\vec{R})],$$

where u, v, and w are constants. Because σ and ϵ are scalars and t_{11} is a component of a second-rank tensor, this result may be extended to arbitrary directions of $\vec{r}$ in the form

$$\sigma_{\vec{r}_1}\sigma_{\vec{r}_2}\sim 1+ur^{\nu_\epsilon}\epsilon(\vec{R})+v\sum_{ij}r_ir_jt_{ij}(\vec{R}). \tag{8}$$

Next differentiate Eq. (8) with respect to y_1 (or y_2) and set $y_1=y_2=0$. Then Eq. (4c) implies

$$D_{-3/2}(\vec{r}_1)D_{1/2}(\vec{r}_2)\sim\frac{u}{2}r^{\nu_\epsilon}\frac{\partial\epsilon(\vec{R})}{\partial Y}+2vrt_{12}(\vec{R})\sim D_{-2}(\vec{R}),\quad D_{1/2}(\vec{r}_1)D_{-3/2}(\vec{r}_2)\sim\frac{u}{2}r^{\nu_\epsilon}\frac{\partial\epsilon(\vec{R})}{\partial Y}-2vrt_{12}(\vec{R})\sim D_2(\vec{R}). \tag{9}$$

There are three possibilities in the interpretation of Eq. (9). If $\nu_\epsilon-1<0$, the $\partial\epsilon/\partial y$ term dominates and $D_{\pm 2}\sim\partial\epsilon/\partial y$; if $\nu_\epsilon-1>0$, the t_{12} term dominates and $D_{\pm 2}\sim\pm t_{12}(R)$; and, finally, if $\nu_\epsilon=1$, both terms contribute to $D_{\pm 2}$ and

$$D_{\pm 2}(\vec{R})\sim\frac{u}{2}\frac{\partial\epsilon(\vec{R})}{\partial Y}\pm 2vt_{12}(\vec{R}). \tag{10}$$

But since the algebra implies that D_2 and D_{-2} behave differently, only the third possibility is tenable; hence $\nu_\epsilon=1$. This determination of the index ν_ϵ implies, via scaling, that the specific-heat index α is equal to zero.

(b) Index for spinor variables.—The combination of operators

$$D_{3/2}(\vec{r}_1)D_{1/2}(\vec{r}_2)-D_{-1/2}(\vec{r}_1)D_{-3/2}(\vec{r}_2) \tag{11}$$

is

$$c(\partial/\partial y)[D_{-1/2}(\vec{r}_1)D_{1/2}(\vec{r}_2)]$$

from Eq. (4c), and is therefore proportional to $\partial a_-(\vec{R})/\partial Y$. From Eqs. (3a) and (7), it is also proportional to $\partial a_+(\vec{R})/\partial X$. Thus,

$$[\partial a_+(\vec{r})/\partial x]+g[\partial a_-(\vec{r})/\partial y]=0,$$

where g is some constant. Application of the same argument with opposite signs for all the subscripts shows that the spinor

$$a(\vec{r})=\begin{pmatrix}a_+(\vec{r})\\ a_-(\vec{r})\end{pmatrix}$$

satisfies

$$\left(\frac{\partial}{\partial x}+g\tau_3\tau_1\frac{\partial}{\partial y}\right)a(\vec{r})=0, \tag{12}$$

where τ_3 and τ_1 are standard Pauli spin matrices. Direct calculations[9] show that at $T=T_c$ the spinors indeed satisfy a first-order differential equation of the form (12) in the scaling limit. Consequently, the derivation of this equation can be considered to be a confirmation of the power of the algebraic method.

According to Eq. (3) the average of the product of spinors $a(\vec{r}_1)a^\dagger(\vec{r}_2)$ is $\tau_1h/r^{2\nu_1}$, where h is a constant and ν_1 is the scaling index for a. By using the rotational properties of real spinors, we can generalize this result to arbitrary $\vec{r}_1-\vec{r}_2$ as

$$\langle a(\vec{r}_1)a^\dagger(\vec{r}_2)\rangle=\frac{h[\tau_1(x_1-x_2)-\tau_3(y_1-y_2)]}{r^{2\nu_1+1}}. \tag{13}$$

But (12) and (13) can only be consistent in two cases: $g=1$, $\nu_1=\frac{1}{2}$ or $g=-1$, $\nu_1=-\frac{1}{2}$. Since $a_\pm$ are bounded, the latter is impossible and we find that $\nu_1=\frac{1}{2}$.

(c) Index for σ.—When $\vec{r}_1$ and $\vec{r}_2$ are on the x axis, Eqs. (3) and (7) imply that, in leading order,

$$\sigma(\vec{r}_1)\mu(\vec{r}_2) \sim \frac{1}{r^{2\nu_{1/2}-\nu_1}} a_+(\vec{R}),$$

where $\nu_{1/2}$ is the scaling index of both μ and σ. The generalization of this result to arbitrary positions is

$$\sigma(\vec{r}_1)\mu(\vec{r}_2) \sim r^{\nu_1-2\nu_{1/2}}[\cos(\theta/2)a_+(\vec{R}) - \sin(\theta/2)a_-(\vec{R})], \quad (14)$$

where θ is the angle between $\vec{r}$ and the x axis. By taking $\partial^2/\partial y_1 \partial y_2$ on Eq. (14) and setting $y_1-y_2=0$, we construct $D_{-3/2}(\vec{r}_1)D_{3/2}(\vec{r}_2) \sim D_3(\vec{R}) \approx 0$. Therefore the leading term must vanish and

$$\frac{\partial^2}{\partial y^2}[r^{\nu_1-2\nu_{1/2}}\cos(\theta/2)]|_{\theta=0}=0,$$

which is only possible if $\nu_1-2\nu_{1/2}=\frac{1}{4}$ so that $\nu_{1/2}=\frac{1}{8}$. Since this is the correct value for the scaling index of the magnetization, we now have found all critical indices by algebraic arguments based upon reduction formulas.

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