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PHYSICAL REALIZATION OF NONCOMPACT DYNAMICAL SYMMETRIES

J. Katriel and G. Adam
Nuclear Research Centre-Negev, P.O.B. 9001, Beer-Sheva, Israel
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The dynamical symmetry of a charged spinless harmonic oscillator in a constant magnetic field is identified. For low fields it is SU(3), for high fields SU(2,1), and for a certain definite intermediate field SU(2) ⊗ H.

The problem of a charged particle in a magnetic field was recently discussed, with new interest in the infinite degeneracy of the continuous eigenenergies.[1] It was also noticed that when a harmonic-oscillator potential is added to the Hamiltonian, one can, with a certain choice of parameters, get a system with an infinitely degenerate discrete spectrum.[2] The extent to which the "accidental" degeneracy of a quantum system is understandable in terms of irreducible representations of the dynamical symmetry group of the corresponding Hamiltonian is of current active interest.[3] Dynamical symmetry groups hitherto considered—e.g., $O(n+1)$ for the n-dimensional Kepler problem, $SU(n)$ for the n-dimensional isotropic harmonic oscillator—are all compact. Noncompact groups which entered the field of dynamical symmetries are the noninvariance groups of these systems. An attempt is made to relate the above-mentioned infinite degeneracy to a noncompact invariance group.

The Hamiltonian for a charged spinless particle in a constant magnetic field $F=2Mc\omega/e$ directed towards the z axis and a potential $\frac{1}{2}M\omega^2z^2$ can be shown to be[2]

$$\mathcal{H}=p^2/2M+M\omega^2r^2/2+\omega L_z. \tag{1}$$

Defining $H=\mathcal{H}/\hbar\omega$ and introducing the notation of second quantization we get

$$H=a_x^\dagger a_x+a_y^\dagger a_y+a_z^\dagger a_z+\tfrac{3}{2}+i(a_y^\dagger a_x-a_x^\dagger a_y). \tag{2}$$

Let

$$A=(a_x+ia_y)/\sqrt{2},\quad A^\dagger=(a_x^\dagger-ia_y^\dagger)/\sqrt{2};$$
$$B=(a_x-ia_y)/\sqrt{2},\quad B^\dagger=(a_x^\dagger+ia_y^\dagger)/\sqrt{2};$$
$$C=a_z,\quad C^\dagger=a_z^\dagger. \tag{3}$$

These operators satisfy boson commutation relations analogous to the Cartesian creation and annihilation operators introduced in Eq. (2). With this notation it follows straightforwardly that

$$H=2B^\dagger B+C^\dagger C+\tfrac{3}{2}. \tag{4}$$

This is seen to be equivalent to a two-dimensional anisotropic harmonic oscillator with $\omega_B=2$, $\omega_C=1$. The dynamical symmetry group of this Hamiltonian was shown to be SU(2).[4] With this observation, any degeneracy associated with the four operators B, $B^\dagger$, C, $C^\dagger$, has been taken into account. It is clear, however, SU(2) being compact and thus having finite-dimensional representations, that no infinite degeneracy has been introduced. The additional degeneracy present is associated with the third coordinate, represented by A and $A^\dagger$. These two operators generate the noncompact Heisenberg group H.[5] As they commute with the four operators generating SU(2) and with the Hamiltonian it follows that the dynamical symmetry group of the Hamiltonian is SU(2) ⊗ H.

For the more general system having a potential $k'(x^2+y^2)+kz^2$ and in a magnetic field directed towards the z axis but of strength $F=2c[(k-k')M]^{1/2}/e$, we get[2]

$$\mathcal{H}=p^2/2M+\tfrac{1}{2}kr^2+eFL_z/2Mc$$

or, using the notation of Eq. (3),

$$H=\alpha A^\dagger A+(2-\alpha)B^\dagger B+C^\dagger C+\tfrac{3}{2}, \tag{5}$$

where $\alpha=1-eF/2Mc\omega$. For the case $F=2Mc\omega/e$ we get $\alpha=0$ which brings us back to Eq. (4). Reversing the magnetic field we get a Hamiltonian $H=2A^\dagger A+C^\dagger C+\frac{3}{2}$ with similar consequences. For $0<\alpha<2$, i.e., $|F|<2Mc\omega/e$, the Hamiltonian, Eq. (5), is that of a three-dimensional anisotropic harmonic oscillator. For $|F| > 2Mc\omega/e$ we get a negative coefficient either of $A^\dagger A$ or of $B^\dagger B$. This Hamiltonian represents a three-dimensional anisotropic harmonic oscillator with negative mass in the direction associated with the negative coefficient. Its energy spectrum spreads discretely from $-\infty$ to ∞. This situation can be understood in the original coordinates as resulting from the magnetic field being strong enough to cause the splitting to be so wide that one gets an infinite number of levels in the negative energy range. It is easily seen, observing the manner in which the splitting occurs, that for rational α in the range $\alpha < 0$ or $\alpha > 2$ infinite degeneracy results. To obtain this result rigorously let

$$\theta^\dagger = (A^\dagger)^n (B^\dagger)^m. \tag{6}$$

The commutator of this operator with the Hamiltonian defined in Eq. (5) is $[H, \theta^\dagger] = [-\alpha(m-n) + 2m]\theta^\dagger$. For $\alpha = 2m/(m-n)$, i.e., any rational α, $[H, \theta^\dagger] = 0$. Starting from any eigenstate, an infinite number of states can be produced by operating successively with $\theta^\dagger$. That $\theta^\dagger|\rangle$ cannot vanish follows from its being a creation operator for quanta in a harmonic oscillator. From the fact that $\theta^\dagger$ commutes with the Hamiltonian it follows that the infinite number of states produced are all degenerate. As the invariance group of a harmonic oscillator is SU(3),[6] it follows immediately that for the system represented by the Hamiltonian Eq. (5) which is a non-positive-definite invariant the appropriate invariance group is the noncompact counterpart SU(2, 1). This is seen most transparently by introducing complex coordinates[7] thus formally identifying $A^\dagger A$ with $|A|^2$, etc., and using the definition of SU(2, 1) given for example by Biedenharn.[8]

A further study of the transition from SU(3) through SU(2)⊗H to SU(2, 1), the associated deformation in the Lie algebra, and, particularly, the implication of some results on energy-level crossing[9] seems to be desirable. A complete identification of the presumably reducible representations of SU(2, 1) in the various situations realized should also be interesting.

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LONGITUDINAL MOMENTUM DISTRIBUTION OF PIONIZATION PRODUCTS*

Hung Cheng†
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

and

Tai Tsun Wu
Gordon McKay Laboratory, Harvard University, Cambridge, Massachusetts 02138

(Received 6 October 1969)

We obtain, on the basis of general principles only, a quantitative formula for the distribution of pionization products in ultrahigh-energy scattering as a function of the longitudinal momentum. Field theory is then used as a model to show the existence of the pionization process, and it also serves as an example in which the formula is satisfied. A highly feasible storage-ring experiment is proposed to test this predicted distribution and hence the underlying principles.

There are two prominent features in cosmic-ray events[1]: (i) two fire balls, and (ii) production of pions of relatively low evergies in the c.m. system. Recently, there has been a great deal of interest in this second feature,[2] commonly referred to as pionization.

We wish to point out in this Letter that quantitative results on the distribution of these pions can be obtained from the thesis[3] that hadrons are extended objects with many internal degrees of