bound requires three Cauchy moments while with the present method the upper bound can be constructed with a single moment (the static polarizability).

For imaginary frequency, we can use the same form of continued factorization of Eq. (4). One can show, by examining the remainder, that $\alpha^{(n)}(i\omega)$ is a lower bound to $\alpha(i\omega)$ for all orders of n. To construct an upper bound with the present method, we can make use of the total number of electrons Z in the atom. If we define

$$Z_n = Z + \sum_{k=1}^{n} a_k \mu_{k-1}, \tag{8}$$

we can show that

$$\beta^{(n)}(i\omega) = \frac{\epsilon_1^2\alpha_0}{\epsilon_1^2+\omega^2} + \frac{\omega^2}{\epsilon_1^2+\omega^2}\left\{\frac{\epsilon\alpha}{\epsilon_2^2+\omega^2} + \frac{\omega}{\epsilon_2^2+\omega^2}\left[\frac{\epsilon_3^2\alpha_2}{\epsilon_3^2+\omega^2} + \frac{\omega^2}{\epsilon_3^2+\omega^2}\left(\cdots + \frac{\omega^2}{\epsilon_n^2+\omega^2}\cdot\frac{Z_n\alpha_n}{Z_n+\alpha_n\omega^2}\right)\cdots\right]\right\} \tag{9}$$

is an upper bound to $\alpha(i\omega)$ for all n. This is because

$$\frac{Z_n\alpha_n}{Z_n+\alpha_n\omega^2} \geqslant \sum_I \frac{f_I^{\,n}}{\epsilon_I^2+\omega^2}, \tag{10}$$

which corresponds to the two-point Slater-Kirkwood-Padé inequality.[5] These bounds are illustrated in Fig. 2 for the hydrogen atom. The convergence rate is about the same as the standard Padé approximant. To tighten either the upper or the lower bound, the Padé method requires two Cauchy moments at a time whereas with the present method we need only one Cauchy moment for each step.

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REMEASUREMENT OF $\Delta E - \mathcal{S}$ IN ATOMIC HYDROGEN*

B. L. Cosens† and T. V. Vorburger‡
Department of Physics, Yale University, New Haven, Connecticut 06520
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The $2^2S_{1/2}$-$2^2P_{3/2}$ separation $\Delta E - \mathcal{S}$ in hydrogen has been measured by a microwave atomic-beam technique. The result obtained from four independent transitions is $\Delta E - \mathcal{S} = 9911.173 \pm 0.042$ MHz. We combine this result with the recent determination of the Lamb shift $\mathcal{S}$ by Robiscoe to obtain a value for the fine-structure splitting ΔE and a value for the fine-structure constant $\alpha^{-1} = 137.0358(5)$.

We have measured $\Delta E - \mathcal{S}$, the splitting between the states $2^2S_{1/2}$ and $2^2P_{3/2}$ in atomic hydrogen (Fig. 1). The experiment is modeled on the original work of Lamb and co-workers.[1] A beam of hydrogen atoms is produced with a single hyperfine component of the $2^2S_{1/2}(m_J = -\frac{1}{2})$ metastable level. The beam is subjected to an rf electric field in a magnetic field of ≈ 400 G oriented parallel to the beam axis. Electric dipole transitions are thereby induced to the b or d levels in the $2^2P_{3/2}$ state (see Fig. 1). We have measured four independent transitions. Table I shows the initial and final states along with the frequency and magnetic field for each.

During a data run, the frequency of the electric field is held constant to 1 ppm, and the magnetic field is swept through the resonance line. A typical experimental resonance is plotted in Fig. 2

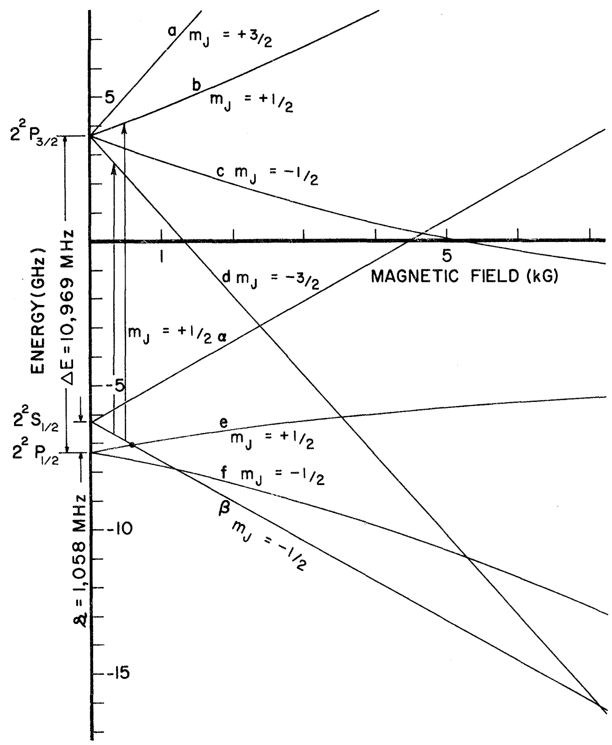

FIG. 1. Zeeman diagram for H, $n=2$. ΔE and $\mathcal{S}$ are shown. The transitions βb and βd are represented by arrows. The β-e crossing is also shown.

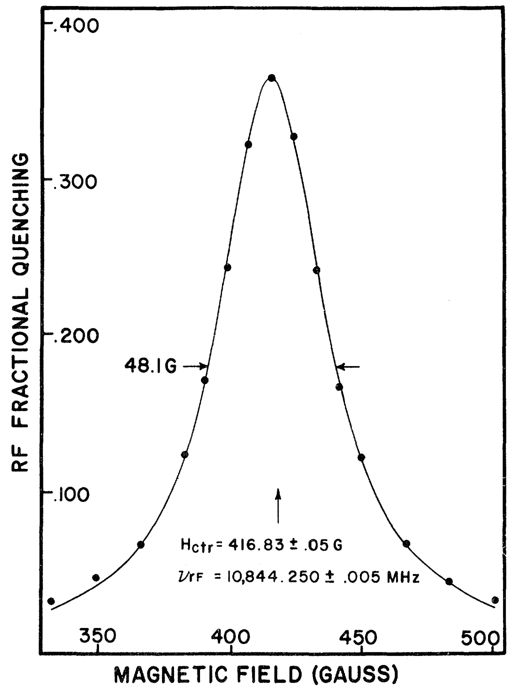

FIG. 2. Resonance curve for run 22. The experimental points have been corrected for α overlap. The theoretical curve has been derived using the Bethe-Lamb theory for the lifetime of the $2S$ state under a perturbing electric field. The line shape has been averaged over a v^4 velocity distribution.

together with the theoretical curve based on the Bethe-Lamb theory. The center of the resonance is determined by measuring the beam quenching at the steepest part of the curve on each side of the resonance. A typical run included 10-20 pairs of these center measurements and yields a determination of the central magnetic field to ≈100 ppm. An extrapolation to zero field is made using a computer diagonalization of Brodsky and Parsons' Hamiltonian[2] for the $n=2$ level of hydrogen. Each of the four transitions yields an independent value of $\Delta E-\mathcal{S}$.

There are a number of corrections which must be applied to the observed resonance centers. Many of these corrections have been explained in previous papers by Lamb et al.,[1] Robiscoe,[3] and Cosens.[4]

The most important corrections arise from overlapping α resonances. The αa transition has a resonant field of 670 G for an rf frequency of 10 840 MHz. This resonance overlaps the βb transition at 400 G (see Table I) and causes an

Table I. Results of the four transitions.

Transition	Initial level $2^2S_{1/2}$	Final level $2^2P_{3/2}$	Approximate frequency (MHz)	Approximate field (G)	Number of centers	$\Delta E-\mathcal{S}$ (MHz)	Random error (MHz)	Systematic error (MHz)
βb^-	$m_J=-\frac{1}{2}$ $m_I=-\frac{1}{2}$	$m_J=+\frac{1}{2}$ $m_I=-\frac{1}{2}$	10 845	417	64	9911.281 ± 0.092	0.074	0.055
βb^+	$m_J=-\frac{1}{2}$ $m_I=+\frac{1}{2}$	$m_J=+\frac{1}{2}$ $m_I=+\frac{1}{2}$	10 844	375	71	9911.144 ± 0.085	0.076	0.039
βd^-	$m_J=-\frac{1}{2}$ $m_I=-\frac{1}{2}$	$m_J=-\frac{3}{2}$ $m_I=-\frac{1}{2}$	9406	336	43	9911.196 ± 0.076	0.073	0.022
βd^+	$m_J=-\frac{1}{2}$ $m_I=+\frac{1}{2}$	$m_J=-\frac{3}{2}$ $m_I=+\frac{1}{2}$	9406	391	46	9911.084 ± 0.084	0.065	0.053
Weighted average						9911.173 ± 0.042		

asymmetry in the βb line shape. Similarly, the αc resonance overlaps βd. We reduce the overlap problem by the following procedure. At each magnetic field we measure the rf beam flop which includes the β and the overlapping α signals. Then we remove the β-state atoms from the beam by applying a dc electric field to induce βe transitions. On this α-only beam the rf beam flop is measured again. This signal is then subtracted from the original $\beta+\alpha$ signal to yield the β resonance. This process will be explained in greater detail in a later paper.

The existence of a velocity distribution in the metastable component of the beam distorts the resonance line shape. This effect was originally discovered by Robiscoe.[3] The size of the correction that must be made depends upon the velocity distribution used. Robiscoe and Shyn have recently shown that the distribution is approximated by a normalized v^4 distribution[5] and depends on the geometry of the apparatus. We have assumed a v^4 distribution in our calculations. The correction does not critically depend on the distribution that is assumed, however, because the βb and βd transitions have opposite field dependencies. Consequently, the final value of $\Delta E-\mathcal{S}$ tends to average out errors in these distortion calculations.

The final results are shown in Table I. The quoted error for each transition is the quadratic sum of the random error of one standard deviation of the mean and the estimated 68% confidence interval for the systematic corrections. The weighted mean result from the four transitions is[6]

$$\Delta E-\mathcal{S}=9911.173\pm0.042\ \text{MHz}.$$

The final quoted error is one standard deviation of the mean. The variance of the four numbers about the mean is 0.040 MHz.

Our final result disagrees with the result of Shyn et al.[7] by ≈1 standard deviation. They have modified their result to 9911.250 ± 0.063 MHz. It disagrees with the result of Kaufman et al.[8] by ≈4 standard deviations. They quote $\Delta E-\mathcal{S}=9911.38\pm0.03$ MHz.

We may calculate the fine-structure splitting ΔE by adding the result for $\Delta E-\mathcal{S}$ to the measurement of $\mathcal{S}$ by Robiscoe.[3] Robiscoe and Shyn have corrected the results for $\mathcal{S}$ to 1057.896 ± 0.063 MHz.[5,9] The value of ΔE is then equal to 10 969.069 ± 0.076 MHz, which agrees with the result of 10 969.13 ± 0.12 MHz obtained by Metcalf, Brandenberger, and Baird.[10]

ΔE and the fine-structure constant α are related by the formula[11]

$$\Delta E=\frac{\alpha^2}{16}c\,\mathrm{Ry}\left[(1+\tfrac{5}{8}\alpha^2)\left(1-\frac{m_e}{m_p}\right)+(g_s-2)\left(1-2\frac{m_e}{m_p}\right)+\frac{2}{\pi}\alpha^3\ln\alpha\right],$$

where c is the speed of light, Ry is the Rydberg wave number, g_s is the electron g factor, and m_e and m_p are the masses of the electron and proton, respectively. Our value for ΔE yields a final result for $\alpha^{-1}=137.0358(5)$. This value agrees with the values obtained from the measurements of the hydrogen ground-state hyperfine splitting (H hfs),[12] and the measurement of $2e/h$ using the ac Josephson effect.[6]

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†Present address: Physics Department, St. Olaf College, Northfield, Minn. 55057.

‡Present address: Physics Department, University of Delaware, Newark, Del. 19711.

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