

or  $(\frac{3}{2}, \frac{1}{2})$  representations. We shall not discuss further here the consequences of assuming mixed representations for the  $A_2$  structure.

<sup>16</sup>Alternatively, by assigning  $\rho$  to  $(0,1)$ ,  $(\pi\rho)$  decays of all three  $A_{2,I}$  ( $I=0,1,2$ ) are  $SU(2) \otimes SU(2)$  allowed. We then invoke the stated decay rule to suppress decay widths for the abnormal members. Note this luxury is not possible were  $A_{2,I}$  assigned the  $(\frac{3}{2}, \frac{1}{2})$  or  $(\frac{1}{2}, \frac{3}{2})$  representation, or a mixture thereof (cf. Ref. 15).

<sup>17</sup>M. Gormley, et al., Phys. Rev. Letters 21, 402 (1968), and 22, 108 (1969).

<sup>18</sup>Like broken  $SU(3)$ , evidence of  $SU(2) \otimes SU(2)$  is expected to be most prominent in mass spectra with decreasing manifestations in decay and four-body processes.

<sup>19</sup>This reaction is  $SU(2) \otimes SU(2)$  allowed if  $\pi$  is assigned to  $(0,1)$  rather than to  $(1,0)$ , or given mixed as-

signments  $(0,1)$  and  $(1,0)$ .

<sup>20</sup>J. Rosner, Phys. Rev. Letters 21, 950 (1968), and 22, 689 (1969); V. Barger and D. Cline, Phys. Rev. 182, 1849 (1969).

<sup>21</sup>Note that despite the near  $\rho^0-\omega^0$  mass degeneracy, they cannot be assigned to the  $(\frac{1}{2}, \frac{1}{2})$  representation because these states have opposite  $G$  parity.

<sup>22</sup>R. D. Tripp, in Proceedings of the Fourteenth International Conference on High Energy Physics, Vienna, Austria, September, 1968 (CERN Scientific Information Service, Geneva, Switzerland, 1968), p. 173.

<sup>23</sup>T. D. Lee and C. N. Yang, Phys. Rev. 122, 1954 (1961).

<sup>24</sup>M. Gell-Mann, Phys. Rev. 106, 1296 (1957); J. Schwinger, Ann. Phys. (N.Y.) 2, 407 (1957); R. P. Feynman, "A Model of Strong and Weak Couplings" (unpublished).

### EXTERIOR THREE-PARTICLE WAVE FUNCTION\*

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It is shown that the wave function for three particles interior to the finite volume where all force ranges overlap completely determines the exterior wave function via the solution of a one-variable integral equation with a compact kernel. A unitary parametrization of the interior wave function therefore would supply the equivalent of a phase-shift analysis for three-particle final states, including the exact description of overlapping resonances throughout the Dalitz plot.

For (central, spin-independent) forces of finite range the exterior wave function for two particles can be uniquely specified by a single phase shift in each angular-momentum state, which parameters also specify the scattering cross section. So far, the corresponding description for three-particle final states has not been constructed. At first sight, the Faddeev<sup>1</sup> decomposition into the three channels which asymptotically contain an interacting pair plus a free particle would seem to solve the problem. However, so long as the interacting pair are within the range of the two-particle force, their energy and momentum are not necessarily connected as they will be in the three-free-particle final state, and they can pick up momentum from the outgoing wave in one of the other two channels (cf. Fig. 1). This produces a nonlocal<sup>2</sup> interaction which falls off only with the inverse distance to the third particle, even if the two-particle forces are of finite range. This in turn produces a singularity in the Faddeev equations which must be removed (e.g., by contour rotation or iteration) before they can be solved. In addition, any strongly interact-

ing system can be expected to have three-body forces where all three-particle ranges overlap, and these must be specified, in addition to the

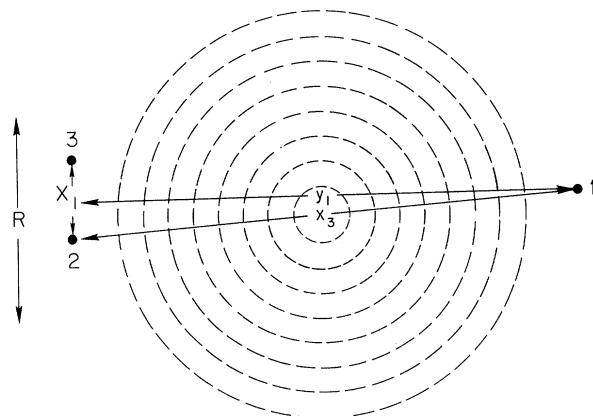


FIG. 1. Scattering in the "three" channel (relative coordinate  $x_3$ ) produces an outgoing wave which can scatter from the particles in the "one" channel (relative coordinate  $x_1$ ) so long as they are within the range of forces,  $R$ . The effect falls off as  $1/y_1$  regardless of the range of forces, and hence is nonlocal.

two-particle forces, before the physical prediction of three-particle cross sections can be achieved. Thus the Faddeev equations provide a dynamical description, but do not provide a means of separating the exterior from the interior wave function. We show below that by reformulating the problem in configuration space, it is possible to make such a separation. Because of the long-range effect described above, the formalism necessarily requires a complete description of the wave functions of the isolated two-particle systems inside the range of the two-particle forces (half off-shell two-particle  $t$  ma-

trix), but still can be made whether or not there are three-particle forces inside the finite volume where the force ranges of all three pairs overlap. Hence, to the extent that one believes a covariant description of the half off-shell two-particle  $t$  matrices (e.g., via the Blankenbecler-Sugar<sup>3</sup> equation), the description could also be extended to the relativistic case. If the forces are known in the interior region, the method also provides nonsingular dynamical equations for the three-body problem.

The wave function  $\Psi(\vec{r}_1, \vec{r}_2, \vec{r}_3)$  for three particles of masses  $m_1, m_2, m_3$  can be re-expressed in terms of the new coordinates

$$\begin{aligned}\vec{R} &= (m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3) / (m_1 + m_2 + m_3), \quad \vec{x}_i = [2m_j m_k / (m_j + m_k)]^{1/2} (\vec{r}_j - \vec{r}_k), \\ \vec{y}_i &= [2m_l (m_j + m_k) / (m_1 + m_2 + m_3)]^{1/2} [-\vec{r}_i + (m_j \vec{r}_j + m_k \vec{r}_k) / (m_j + m_k)],\end{aligned}\quad (1)$$

( $i, j, k$  cyclic on 1, 2, 3) and decomposed into Faddeev channels and radial and angular parts according to

$$\Psi_J^M = e^{i\vec{P} \cdot \vec{R}} \sum_{s=1}^3 \sum_{l\lambda} \frac{U_{l\lambda}^{Js}(x_s, y_s)}{x_s y_s} Y_{Jl\lambda}^M(\theta_{\vec{x}_s}, \varphi_{\vec{x}_s}, \theta_{\vec{y}_s}, \varphi_{\vec{y}_s}), \quad (2)$$

where  $\vec{P}$  is the total momentum and the  $Y_{Jl\lambda}^M$  are the two-direction spherical harmonics as defined by Blatt and Weisskopf.<sup>4</sup> If the interactions are due to central, spin-independent potentials  $V^i(|\vec{r}_j - \vec{r}_k|)$  and  $W^i(x_i) = V_i([m_j + m_k]^{1/2} x_i)$ , the radial wave functions  $U_{l\lambda}^{Ji}(x, y)$  are uniquely<sup>5</sup> specified for each value of  $J$  (which index we now drop) by the dynamical equations

$$\begin{aligned}\left[ \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} - \frac{l(l+1)}{x_i^2} - \frac{\lambda(\lambda+1)}{y_i^2} + z - W^i(x_i) \right] U_{l\lambda}^i(x_i, y_i) \\ = W^i(x_i) \sum_{s=j,k} \int_{|\varphi_i - \mu_{is}|}^{\min(\varphi_i + \mu_{is}, \pi - \varphi_i - \mu_{is})} d\varphi_s \sum_{l'\lambda'} K_{l\lambda l'\lambda'}^{is}(\varphi_i, \varphi_s) U_{l'\lambda'}^{is}(r \cos \varphi_s, r \sin \varphi_s) \\ \equiv W^i(x_i) S_{l\lambda}^i(x_i, y_i),\end{aligned}\quad (3)$$

$$r = (x_i^2 + y_i^2)^{1/2}, \quad \varphi_i = \tan^{-1} \frac{y_i}{x_i}, \quad \cos \mu_{is} = \left[ \frac{2m_l m_s}{(m_l + m_s)(m_s + m_l)} \right]^{1/2} \quad (s=j, s'=k \text{ or } s=k, s'=j),$$

$$z = E_{LAB} - P^2 / 2(m_1 + m_2 + m_3).$$

The geometrical factor coupling in the two other channels is, explicitly,

$$K_{l\lambda l'\lambda'}^{is}(\varphi_i, \varphi_s) = \frac{8\pi^2}{(2J+1)^{1/2} \sin 2\mu_{is}} \sum_M Y_{Jl\lambda}^{M*}(\xi, 0, \xi + \xi_i, 0) Y_{Jl'\lambda'}^M(\xi + \xi_{is}, 0, \xi + \xi_{is} + \xi_s, 0),$$

with

$$\begin{aligned}\cos \xi_i &= (\cos 2\mu_{is} \cos 2\varphi_i - \cos 2\varphi_s) / \sin 2\mu_{is} \sin 2\varphi_i, \\ \cos \xi_{is} &= (\sin \mu_{is} \sin \varphi_i \cos \xi_i - \cos \mu_{is} \cos \varphi_i) / \cos \varphi_s, \\ \cos \xi_s &= (\cos 2\mu_{is} \sin 2\varphi_i \cos \xi_i + \sin 2\mu_{is} \cos 2\varphi_i) / \sin 2\varphi_s,\end{aligned}\quad (4)$$

and is independent of the angle  $\xi$  which  $\vec{x}_i$  makes with some arbitrary axis fixed in the plane of the triangle. For the states of zero total and relative angular momentum,  $K$  is just  $1/\sin 2\mu_{is}$  and for three particles of the same mass,  $\mu_{is}$  is  $\pi/3$ .

Since the two-particle forces  $W^i$  are assumed known, we can construct the Green's function for the left-hand side of Eq. (3) in terms of product wave functions in  $x$  and  $y$ ,  $u_p^{Ji}(x) f_\lambda(qy)$ , which vanish at

$x=0$  or  $y=0$  and which we choose to normalize asymptotically to  $u_p^{II}(x) - \sin(px + \delta_p^{II} - l\pi/2)$ ,  $f_\lambda(qy) - \sin(qy - \lambda\pi/2)$  [i.e.,  $f_\lambda(qy) = qy j_\lambda(qy)$ ]. They therefore have the orthonormality properties

$$\frac{2}{\pi} (\int + \sum) dp u_p^{II}(x) u_p^{II}(x') = \delta(x-x'), \quad \frac{2}{\pi} \int_0^\infty dq f_\lambda(qy) f_\lambda(qy') = \delta(y-y'), \quad (5)$$

where the symbol  $(\int + \sum)$  is written to remind us that if  $W^I$  is strong enough to support bound states at  $p = i\gamma$ , these discrete terms must also be included. Since, when this Green's function is applied to the source term on the right, we encounter  $u_p^{II}(x') W^I(x')$ , we can eliminate explicit reference to the potential in favor of the half off-shell  $t$  matrix by the relation

$$u_p^{II}(x) W^I(x) = -\frac{2}{\pi} p \exp(-i\delta_p) \int_0^\infty dk k f_I(kx) t_I^I(k, p; p^2),$$

with

$$t_I^I(p, p; p^2) = \exp(i\delta_p^{II}) \sin \delta_p^{II} / p \equiv \tau(p). \quad (6)$$

If the right-hand side of Eq. (3) were bounded in both  $x$  and  $y$ , the Green's function solution

$$U_{I\lambda}^I(x, y) = X_{I\lambda}^I(x, y) + \frac{4}{\pi^2} (\int + \sum) dp \int_0^\infty dq \int_0^\infty dx' \int_0^\infty dy' \frac{u_p^{II}(x) f_\lambda(qy) u_p^{II}(x') f_\lambda(qy')}{z + i\epsilon - p^2 - q^2} W^I(x') S_{I\lambda}^I(x'y') \quad (7)$$

would have the exterior representation

$$U_{I\lambda}^{I \text{ ext}}(x, y) = X_{I\lambda}^I(x, y) + (\int_0^\infty + \sum) dq e_I[(z - q^2)^{1/2} x] f_\lambda(qy) F_{I\lambda}^I(q), \\ e_I[(z - q^2)^{1/2} x] = i(z - q^2)^{1/2} x h_I^{(1)}[(z - q^2)^{1/2} x] - \exp[i(z - q^2)^{1/2} x - l\pi/2], \quad (8)$$

with

$$F_{I\lambda}^I(q) = \int_0^\infty dk k t[k, (z - q^2)^{1/2}; z - q^2] \iint_{\text{interior}} dx dy f_I(kx) f_\lambda(qy) S_{I\lambda}^I(x, y). \quad (9)$$

Clearly, the  $F_{I\lambda}^I(q)$  are simply proportional to the Faddeev  $T^I$  in the  $Jl\lambda$  representation, and include elastic scattering and rearrangement collisions via the discrete terms in the sum. Unfortunately, the source term, although bounded in  $x$  if  $W^I(x)$  has a finite range, falls off only as  $1/y$  for reasons discussed in the first paragraph and illustrated in Fig. 1.

The key to a separation of Eq. (3) into exterior and interior parts is the observation that the limits of integration of  $\varphi_s$  plus the assumption that  $W^I$  vanishes for  $x > R$  for all  $i$  is sufficient to limit the region for  $x_s < R$  in which  $U_{I\lambda}^S(x_s, y_s)$  need be known to compute  $S(x, y)$  to the finite domain

$$0 < x_s < R, \quad 0 < y_s < (R + x_s \cos \mu_{I_S}) / \sin \mu_{I_S}, \quad (10)$$

while in the contribution coming from  $x_s > R$  [which lies in the strip bounded by  $y_s = (R \pm x_s \cos \mu_{I_S}) / \sin \mu_{I_S}$ ], it can be computed from the one-variable representation given in Eq. (8). Hence, if we assume the wave function known in this interior region, for example, in terms of some complete set  $A_n(x, y)$  over this finite domain, i.e.,

$$U_{I\lambda}^I(x, y) = X_{I\lambda}^I(x, y) + \theta(R-x) \sum_n a_n A_n(x, y) + \theta(x-R) (\int + \sum) dq e_I[(z - q^2)^{1/2} x] f_\lambda(qy) F_{I\lambda}^I(q). \quad (11)$$

The  $F_{I\lambda}^I(q)$  can be determined by solving the one-variable integral equation

$$F_{I\lambda}^I(q) = \tilde{X}_{I\lambda}^I(q) + \tilde{X}_{I\lambda}^{IB}(q) + \sum_n a_n X_{I\lambda}^{In}(q) + (\int + \sum) dp \sum_{S'I'\lambda'} Q_{I\lambda I\lambda'}^{IS}(q, p) F_{I'\lambda'}^S(p). \quad (12)$$

If the resolvent for  $Q(q, p)$  exists, application to the inhomogeneous term in Eq. (12) immediately gives  $T^I$  in terms of known functions with coefficients  $a_n$ . These known functions are completely determined by the two-body half off-shell  $t$  matrices and the complete set  $A_n(x, y)$ , so the  $a_n$  are analog of phase shifts for the three-particle system.<sup>6</sup> Further, if this exterior representation is used in Eq. (11) and is reinserted in Eq. (3), we obtain an equation for  $U_{I\lambda}^I(x, y)$  over the finite domain [Eq. (10)]. If there are three-body forces, these must be explicitly introduced into Eq. (3) at this point. If the interior wave function can be expanded in terms of the same complete set which suffices to represent the resolvent kernel for Eq. (12), this will give<sup>7</sup> a convergent matrix equation for the  $a_n$ , but the re-

quirements which will allow this have not been established. It remains only to show that  $Q(q, p)$  falls off more rapidly than  $\text{const}/(qp)^{1/2}$  for large  $q$  and  $p$ . Explicitly,

$$Q_{I\lambda I'\lambda'}^{IS}(q, p) = \int_0^\infty dk kt_i^i(k, (z-q^2)^{1/2}; z-q^2) G_{I\lambda I'\lambda'}^{IS}(k, q, p), \quad (13)$$

where  $G$  is a purely geometrical factor given by

$$G_{I\lambda I'\lambda'}^{IS}(k, q, p) = \int_0^\infty r dr \int_0^{\pi/2} d\varphi_i \int_{|\mu_{IS}-\varphi_i|}^{\min(\mu_{IS}+\varphi_i, \pi-\mu_{IS}-\varphi_i)} d\varphi_s K_{I\lambda I'\lambda'}^{IS}(\varphi_i, \varphi_s) \theta(r \cos \varphi_s - R) \\ \times f_I(kr \cos \varphi_i) f_{\lambda'}(qr \sin \varphi_i) e_{I'}[(z-p^2)^{1/2} r \cos \varphi_s] f_{\lambda'}(pr \sin \varphi_s). \quad (14)$$

The factor  $K$  is a product of spherical harmonics of angles in the physical range, and so can only improve the convergence of the integrals and can be safely ignored. If we perform the  $k$  integration over the off-shell extension of  $t$  first, the fact that this comes from the difference between the wave function and its asymptotic form<sup>3</sup> ensures that it will be bounded by a factor proportional to  $1/(k^2 + \beta^2)$  with  $\beta \sim 1/R$ , and the integral of this times  $kf_I(kx)$  will be bounded by something proportional to  $e^{-\beta x}$ , which is no surprise if we look at the left-hand side of Eq. (6). Hence (ignoring bounded factors), the  $q$  dependence will be determined by  $t((z-q^2)^{1/2}, (z-q^2)^{1/2}; z-q^2)$ , which falls off at least as rapidly as  $\text{const}/iq$  for large  $q$ ; [it is also easy to see from other ways of writing  $Q(q, p)$  that there is no difficulty at  $q^2 = z$ ]. Similarly, for large  $p$ , the exponential term becomes  $\exp[-(p^2-z)^{1/2} r \cos \varphi_s]$ , and an asymptotic behavior at least as rapidly decreasing as  $\text{const}/p$  is also guaranteed once we note that  $r \cos \varphi_s$  is bounded from below by  $R$ . Hence  $Q_{I\lambda I'\lambda'}^{IS}(p, q) \leq C/qp$ , which guarantees the existence of a resolvent kernel for Eq. (12).

It is also important to note that if we make the decomposition<sup>3</sup>  $t(k, (z-q^2)^{1/2}; z-q^2) = \tau(z-q^2)^{1/2} \times f_{(z-q^2)^{1/2}}(k)$  in Eqs. (13) and (14), we can investigate in which kinematic regions the geometrical factor will make the result sensitive to the off-shell extension  $f$ , and where the resolvent kernels will depend primarily on the on-shell dependence  $\tau(z-q^2)^{1/2}$ . This will determine where the optimum regions lie for determining the on-shell factor for unstable systems (e.g.,  $\pi-\pi$  phase shifts in  $\pi N \rightarrow \pi\pi N$  final states), and where to investigate off-shell behavior for systems in which the on-shell behavior is known (e.g., the three-nucleon system). As developed above, the analysis is nonrelativistic but since the final equation depends only on the half off-shell  $t$  matrices, the external analysis can be used in relativistic systems to the extent that one has confidence in covariant definitions of the off-shell extension of two-particle  $t$  matrices (e.g., via the

Blankenbecler-Sugar<sup>3</sup> equation). Hence, it can be immediately applied to problems of overlapping resonances in the Dalitz plot and the determination of elementary-particle resonance parameters; note that there is no double counting, and all relative phases are explicitly given. This application, and the inclusion of spin, does not affect the compactness proof given above, but is obviously too complicated to be developed in a short article.

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<sup>1</sup>L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [translation: Soviet Phys.-JETP **12**, 1014 (1961)].

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<sup>3</sup>R. Blankenbecler and R. Sugar, Phys. Rev. **142**, 1051 (1966).

<sup>4</sup>J. M. Blatt and V. F. Weisskopf, Theoretical Nuclear Physics (John Wiley & Sons, Inc., New York, 1952), p. 790.

<sup>5</sup>H. P. Noyes and H. Fiedeldey, in Three-Particle Scattering in Quantum Mechanics, edited by J. Gillespie and J. Nuttall, (W. A. Benjamin, Inc., New York, 1968), p. 195; for the earlier corresponding reduction of the Faddeev equations for two variables, see A. Ahmazadeh and J. A. Tjon, Phys. Rev. **139**, B1085 (1965) and T. Osborn and H. P. Noyes, Phys. Rev. Letters **17**, 215 (1966).

<sup>6</sup>Rather, they will become so once the unitarity conditions on  $T^f$  are imposed; I am indebted to Leonardo Castillejo for pointing out this essential step.

<sup>7</sup>Again I am indebted to Leonardo Castillejo for this remark.

<sup>8</sup>H. P. Noyes, Phys. Rev. Letters 15, 538 (1965).

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ERRATA

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DIELECTRIC CONSTANT AND MOBILITY OF A DOPED ZERO-GAP SEMICONDUCTOR. L. Liu and E. Tosatti [Phys. Rev. Letters 23, 772 (1969)].

Equation (4) on p. 773 should read,

$$\mu = |e|(\hbar k_F n A)^{-1}.$$

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The following citation should be added to footnote 3: R. L. Anderson, R. Prepost, and B. H. Wiik, Phys. Rev. Letters 22, 651 (1969).