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SATURATION OF NONLINEAR EXPLOSIVE INSTABILITIES*

C. T. Dum and R. N. Sudan
Laboratory of Plasma Studies, Cornell University, Ithaca, New York 14850
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The saturation of linear or nonlinearly (explosively) unstable high-frequency, electrostatic flute modes in a mirror-confined plasma is discussed. The mechanism proposed is the perturbation of particle orbits by the wave electric fields.

It has been shown that nonlinear interactions between positive- and negative-energy waves may give rise to explosive instabilities, i.e., wave amplitudes which become infinite in a finite time.[1-4] The question then arises as to what mechanism will ultimately limit wave growth.

The explosive interaction between a single triplet of large-amplitude waves which have a frequency mismatch $\Delta\omega = \omega_{\vec{k}}^r + \omega_{\vec{k}'}^r + \omega_{\vec{k}''}^r \neq 0$ occurs only when the initial amplitudes exceed a threshold proportional to $|\Delta\omega|$.[4] Higher-order interactions of the triplet produce amplitude-dependent shifts in the mode frequencies. If these shifts become sufficiently large, wave growth will terminate.[5]

In the case of interactions between waves of a wide frequency spectrum, frequency shifts will probably not lead to saturation unless they change (at high field levels) the dispersion relation from the decay type to the nondecay type for which the resonance conditions cannot be met. The opposite case of nonlinear frequency shifts which lead to the fulfillment of resonance conditions has been considered by several authors.[6,7]

A difficulty in predicting explosive interactions of many waves and stabilization by higher order terms arises from the fact that the net growth rate is the result of a competition between energy gain in an explosive interaction and the energy drain due to spreading of wave energy in nonexplosive interactions. This situation prevails to every order in conventional perturbation theory, with the number of possible interactions increasing at the same time.

For electrostatic flute modes ($k_\parallel = 0$), which we shall consider below, it has been shown that the possibility of interactions between positive- and negative-energy waves is connected with the loss-cone (non-Maxwellian) nature of the distribution function, and that the crossover from positive to negative wave energy occurs near multiples of the cyclotron frequency.[2,4] The microscopic nature of these interactions suggests that the perturbation of particle orbits by the electric field may ultimately lead to saturation of explosive flute instabilities. To change the nature of the wave modes, e.g., from negative to positive energy presumably requires higher electric field amplitudes.

A theory for the nonlinear stabilization of high-frequency, finite-Larmor-radius instabilities by orbit perturbations has been developed recently for the case of a broad wave spectrum.[8] The perturbed particle motion is then a Brownian motion in the random electric field. We wish to apply this theory to the explosive flute instabilities of mirror-confined plasmas.

Bursts of radiation observed in mirror machines are usually correlated with particle ejection. This is also suggestive of the stabilization mechanism discussed here, since this mechanism is directly related to a large nonlinear enhancement of the diffusion coefficients. It is also conceivable that the enhanced interaction of the waves with the background plasma may lead to a relaxation of the plasma, perhaps producing repetitive bursts of the instability.

The dispersion relation for electrostatic flute modes ($k_{\parallel}=0$) is[8]

$$\epsilon(\vec{k},\omega)=1+\sum\nolimits_j \epsilon_j(\vec{k},\omega)=1-\sum_j \frac{\omega_j^2}{k^2}\int d\vec{v}\left[1-\sum_{m=-\infty}^{\infty} J_m^2\left(\frac{k_\perp v_\perp}{\Omega_j}\right)\frac{\omega}{\omega-m\Omega_j+i\Delta\omega_{\vec{k}}^{(j)}}\right]\frac{1}{v_\perp}\frac{\partial f_j}{\partial v_\perp}. \tag{1}$$

Expression (1) differs from the corresponding linear expression[9] only by the resonance broadening $\Delta\omega_{\vec{k}}^{(j)}$, which arises from the random motion of particles across the magnetic field. $\Delta\omega_{\vec{k}}^{(j)}$ is expressed in terms of an effective transverse diffusion coefficient D,

$$\Delta\omega_{\vec{k}'}^{(j)}=k_\perp'^2 D(\vec{v})=\frac{k_\perp'^2c^2}{B^2}\sum_{\vec{k}}|E_{\vec{k}_\perp}|^2\sum_{m=-\infty}^{\infty}F_m\left(\frac{k_\perp v_\perp}{\Omega_j}\right)\frac{\Delta\omega_{\vec{k}}^{(j)}+\gamma_{\vec{k}}}{(\omega_{\vec{k}}-m\Omega_j)^2+[\Delta\omega_{\vec{k}}^{(j)}+\gamma_{\vec{k}}]^2}, \tag{2}$$

where $F_m(x)=\frac{1}{4}[J_{m-1}^2(x)+2J_m^2(x)+J_{m+1}^2(x)]$ and $\gamma_{\vec{k}}$ is the actual nonlinear growth rate which vanishes when stabilization is achieved.

The broadening results in a nonlinear damping which for small $\Delta\omega_{\vec{k}}$ is given by

$$\Delta\gamma_{\vec{k}}=-\left(\frac{\partial\epsilon}{\partial\omega}\right)^{-1}\sum_j\frac{\omega_j^2}{k^2}\int d\vec{v}\,\frac{1}{v_\perp}\frac{\partial f_j}{\partial v_\perp}\sum_{m=-\infty}^{+\infty}J_m^2\left(\frac{k_\perp v_\perp}{\Omega_j}\right)\frac{\Delta\omega_{\vec{k}}^{(j)}(\vec{v})\omega}{(\omega-m\Omega_j)^2}\bigg|_{\omega=\omega_{\vec{k}}}. \tag{3}$$

If, e.g., $\omega_{\vec{k}}\approx\Omega_j$, then $\gamma_{\vec{k}}\approx\langle\Delta\omega_{\vec{k}}^{(j)}\rangle_{\vec{v}}$ with the velocity average defined by (3).

The actual growth rate is reduced to

$$\gamma_{\vec{k}}=\gamma_{\vec{k}}^0-\Delta\gamma_{\vec{k}}, \tag{4}$$

where $\gamma_{\vec{k}}^0$ is the growth rate resulting from all other linear or nonlinear processes. In the case of nonlinear instabilities, $\gamma_{\vec{k}}^0$ depends on the wave amplitudes, as does $\Delta\gamma_{\vec{k}}$. The saturation amplitude ($\gamma_{\vec{k}}=0$) may be determined from the intersection of $\gamma_{\vec{k}}^0$ and $\Delta\gamma_{\vec{k}}$, or $\gamma_{\vec{k}}^0$ and $\Delta\tilde{\gamma}_{\vec{k}}$ versus amplitude, where $\Delta\tilde{\gamma}_{\vec{k}}$ and $\Delta\tilde{\omega}_{\vec{k}}$ are computed by formally setting $\gamma_{\vec{k}}=0$ in (2) (see Fig. 1).

Estimates of the field dependence of $\gamma_{\vec{k}}^0$ have been given in Ref. 4. The field dependence of $\Delta\gamma_{\vec{k}}$ has been discussed in Ref. 8. We see from (2) that $\Delta\tilde{\omega}_{\vec{k}}$ is zero until the electric field exceeds a threshold determined by ($\gamma_{\vec{k}}=0$)

$$1=\frac{c^2}{B^2}\sum_{\vec{k}}|E_{\vec{k}}^c|^2\sum_{m=-\infty}^{\infty}F_m\left(\frac{k_\perp v_\perp}{\Omega}\right)\frac{k_\perp^2}{(\omega_{\vec{k}}-m\Omega)^2}. \tag{5}$$

Because of the rapid increase in the nonlinear damping when the threshold field level is exceeded, the problem is essentially reduced to the evaluation of the threshold field level. If this level turns out to be sufficiently small, then the mechanism described here is a plausible stabil-

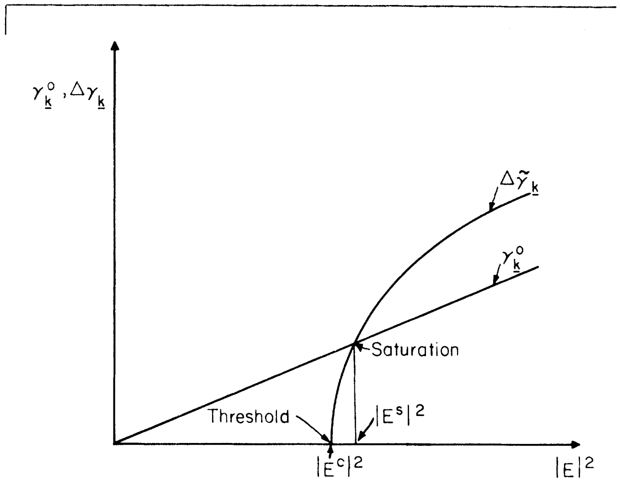

FIG. 1. Nonlinear growth rate $\gamma_{\vec{k}}^0$ of explosive instability and nonlinear damping $\Delta\tilde{\gamma}_k$ versus electric field energy (schematic).

ization mechanism.

Following Ref. 4 we consider high-frequency flute modes for which $(k_\perp v_i/\Omega_i)^2\gg 1\gg(k_\perp/v_e/\Omega_e)^2$ and $\Omega_e\gg\omega\gg\Omega_i$. Using the dispersion relation $\epsilon_{\text{ion}}(\vec{k},\omega_{\vec{k}})=-(1+\omega_e^2/\Omega_e^2)$, the electric field energy density may be related to the charge-density perturbations $n_{\vec{k}}$ by

$$|E_{\vec{k}}|^2/4\pi nT=(k^2\lambda_D^2)^{-1}(1+\omega_e^2/\Omega_e^2)^{-1}|n_{\vec{k}}/n|^2. \tag{7}$$

Since many harmonics $m\Omega_i$ contribute to (1) and (2) it is convenient to sum the series of Bessel functions[2]

$$H(x,\bar{\omega})\equiv\sum_{m=-\infty}^{+\infty}\frac{J_m^2(x)}{(\bar{\omega}-m)^2}=-\frac{\partial}{\partial\nu}\frac{\pi}{\sin\pi\nu}J_\nu(x)J_{-\nu}(x)\bigg|_{\nu=\bar{\omega}}. \tag{8}$$

With (7) and (8) the threshold condition (5) may be written

$$\left(\frac{n_1^c}{n}\right)^2=\frac{(\Omega_i^2/\omega_i^2+m_e/m_i)^2}{\langle H(v_\perp)\rangle}, \tag{9}$$

where $(n_1^{\,c})^2 = \sum_{\vec{k}} |n_{\vec{k}}^{\,c}|^2$ and

$$\langle H(v_\perp)\rangle = \sum_{\vec{k}} S_{\vec{k}} \tfrac{1}{4}[H(x_{\vec{k}}, \bar{\omega}_{\vec{k}}-1) + 2H(x_{\vec{k}}, \bar{\omega}_{\vec{k}}) + H(x_{\vec{k}}, \bar{\omega}_{\vec{k}}+1)], \tag{10}$$

where $S_{\vec{k}} = |n_{\vec{k}}|^2 / \sum_{\vec{k}} |n_{\vec{k}}|^2$ is the normalized spectrum of density perturbations, $\bar{\omega}_{\vec{k}} = \omega_{\vec{k}}/\Omega_i$, and $x_{\vec{k}} = k_\perp v_\perp / \Omega_i$. The expressions for $\Delta\omega_{\vec{k}}$ and $\Delta\gamma_{\vec{k}}$ may be treated similarly. Equation (9) is valid for any electrostatic flute mode. We can apply Debye's asymptotic expansions for large $\bar{\omega}$ and fixed $z = \bar{\omega}/x$ to (10) in view of the dispersion relation[4] which relates $\cot(\pi\bar{\omega})$ to $\omega/k_\perp v_i$.

The threshold density perturbation may thus be estimated as

$$(n_1^{\,c}/n)^2 \approx [(\Omega_i/\omega_i)^2 + m_e/m_i]^2 \langle 1/\bar{\omega}_{\vec{k}}^2\rangle_{\vec{k}}^{-1}, \quad (\omega/k_\perp v_\perp) \gg 1, \ \bar{\omega} \gg 1, \tag{11}$$

and

$$(n_1^{\,c}/n)^2 \approx [(\Omega_i/\omega_i)^2 + m_e/m_i]^2 \langle \{\pi[\cot^2(\pi\bar{\omega}) + 1] - (\bar{\omega}/x)^2 (1/\bar{\omega}) \cot(\pi\bar{\omega})\}/x\rangle_{\vec{k}}^{-1}, \tag{12}$$

for $\omega/k_\perp v_\perp \ll 1$, $\bar{\omega} \gg 1$. The dispersion relation may be used to express $\cot(\pi\bar{\omega})$ in terms of $\omega/k_\perp v_i$ before averaging over the spectrum. From (11), (12), and (7) we obtain approximately $[\cot^2(\pi\bar{\omega}) \approx 1]$

$$\begin{aligned} |E^c|^2/4\pi n T = (cE/Bv_i)^2(\Omega_i/\omega_i)^2 &\approx (\Omega_i/\omega_i)^2(\omega/k_\perp v_i)^2, \quad \omega/k_\perp v_i \gg 1, \\ &\approx (\Omega_i/\omega_i)^2 (2\pi x)^{-1}, \quad \omega/k_\perp v_i \ll 1, \end{aligned} \tag{13}$$

where typical numbers, e.g., $\omega_i^2/\Omega_i^2 = 25$, $x = k_\perp v_i/\Omega_i = 1$, $\omega/\Omega_i = 3$, are to be used on the right-hand side of (13). Stabilization of short-wavelength (negative-energy) modes is seen to occur first.

The decrease of the saturation wave energy with decreasing Ω_i^2/ω_i^2 and the threshold-like behavior of the saturation mechanism predicted here have been observed in computer experiments,[10] although these were done mostly for a few linearly or nonlinearly unstable waves.[11] Saturation occurred abruptly when the electric field reached $cE/B \sim v_i$, independent of the growth rate [cf. (13)]. The observed relaxation of the (delta-function) distribution function (velocity spread) is inhibited in a finite-length mirror machine by the loss of particles scattered into the loss cone. The explosive instabilities treated in Ref. 4 assumed already a broad distribution function $f \sim v_\perp^2 \exp(-v_\perp^2/2v_i^2)$ for which the plasma is linearly stable with respect to flute modes.[12] The saturation mechanism discussed here applies, however, to both linearly and nonlinearly unstable flute modes.

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