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MOBILITY OF AN IMPURITY IN A FERMI LIQUID

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The mobility of ions in liquid He^3 is investigated by a method which avoids the assumption that ions recoil freely in collisions with Fermi quasiparticles. The temperature variation of the mobility is found to be much less than that previously predicted, and improved agreement with experiment is obtained.

The damping force experienced by an impurity of atomic dimensions moving through a Fermi liquid (e.g., ions in He^3 under the influence of an electric field) is mainly due to collisions with quasiparticles if the temperature is sufficiently low. Previous theories of the mobility in this regime[1-3] involve a characteristic temperature $T_0 = k_F^2/2Mk_B$ (k_F, M, and k_B are the Fermi momentum of the liquid, the impurity effective mass, and Boltzmann's constant, respectively, and $\hbar = 1$), at which the mobility temperature dependence is predicted to change from T^{-2} at low temperatures to approximately T^0 at higher temperatures. This prediction is not supported by experiment.[4] With negative ions in He^3 the mobility at low temperatures (below $\frac{1}{2}T_0$ in the most favorable case) is only weakly temperature dependent, while with positive ions the low-temperature behavior (down to $\sim\frac{1}{4}T_0$) resembles $T^{-1/3}$ rather than T^{-2}. The above theories treat the recoil of the impurity during a collision by ascribing to it a definite effective mass. In the following, we describe a theory in which the recoil is treated in a more fundamental manner in terms of the Van Hove scattering function, and the discrepancy between theory and experiment referred to above is largely resolved.

If the impurity were rigid and fixed in position, all scattering from it would be elastic and governed by a differential cross section $\sigma(\theta)$. (The Fermi velocity is assumed to be large compared with all other relevant velocities, so that only the scattering angle θ enters.) In fact the impurity is not fixed and the scattering is inelastic. We write the inelastic scattering cross section per unit solid angle as $\sigma(\theta)S_{\vec{v}}(\vec{K}, \omega)$, where $\vec{K}$ and ω are the momentum and energy transfer, respectively, and $\vec{v}$ is the mean drift velocity of the impurity (under the influence of a fixed external force). $S_{\vec{v}}(\vec{K}, \omega)$ is taken to be the usual scattering function[5,6] which enters into the analogous problem of inelastic neutron scattering,[7] and gives the spectrum of the energy absorbed when the impurity is instantaneously given momentum K. With these assumptions, the rate at which momentum is transferred to the quasiparticles has the form

$$\frac{3n}{4\pi k_F}\iiiint d\Omega_i d\epsilon_i d\Omega_f d\epsilon_f (\vec{k}_f - \vec{k}_i)\sigma(\theta)S_{\vec{v}}(\vec{k}_i - \vec{k}_f, \epsilon_i - \epsilon_f)f(\epsilon_i)[1 - f(\epsilon_f)], \tag{1}$$

where the suffixes i and f appended to the energy and momentum variables refer to the initial and final states, respectively, of a scattered quasiparticle, $f(\epsilon)$ is the equilibrium Fermi factor, and n is the particle density of the fluid.

The v dependence of S_v is obtained by assuming that the only effect of a fixed external force is to superimpose upon the random motion of the impurity a uniform velocity v. Then by Galilean transformation, $S_{\vec{v}}(\vec{K}, \omega) = S_0(K, \omega + \vec{v}\cdot\vec{K})$. An expression for the static mobility μ_s (defined here as velocity/force) may now be obtained by expanding (1) to first order in v. The result is written first in terms of the symmetric combination

$$S_0(K, \omega) + S_0(K, -\omega) = (1 + e^{-\beta\omega}) S_0(K, \omega)$$

(from detailed balance), and then in terms of $F_0(K, t)$, the Fourier transform of $S_0(K, \omega)$. In terms of a dimensionless time variable $\tau = t/\beta$ ($\beta = 1/k_B T$), the final result can be written (with $K = 2k_F \sin\frac{1}{2}\theta$)

$$\mu_s^{-1} = \tfrac{1}{2}\pi^2 n k_F \int_0^\pi d\theta \sin\theta(1-\cos\theta)\sigma(\theta) \int_{-\infty}^{\infty} d\tau \operatorname{sech}^2 \tfrac{1}{2}\pi\tau \, F_0(K, \beta\tau). \quad (2)$$

$F_0(K, t)$ is related to the motion of the impurity over an interval t, being equal to the thermal average of $\exp[-i\vec{K}\cdot\vec{R}(0)]\exp[i\vec{K}\cdot\vec{R}(t)]$, where $\vec{R}(t)$ is the Heisenberg operator for the position of the impurity at time t.

References 1-3 used a collision approach similar to the present one but took into account conservation of energy during a collision by treating the impurity as a quasiparticle with a definite energy-momentum relation governed by an effective mass M. The present theory takes into account in a more general way the effect of the surrounding fluid on the recoil of the impurity. To discuss the relationship between the theories further, we remark that the previous results can be obtained as a special case of (1) by using an $S_v(K, \omega)$ derived using the effective-mass approximation. Furthermore, in these theories an assumption may be made[2] analogous to that used here in deriving (2) from (1), regarding the v dependence of the impurity-momentum distribution function; the mobility expression resulting is a special case of (2). Avoiding this assumption (as in Ref. 1) only slightly affects the results obtained. The previous theories are thus in effect approximations based on using in (2) not the actual F, but that for an ensemble of free particles of mass M (with a velocity distribution corresponding to thermal equilibrium at the temperature concerned).

The important values of τ contributing to (2) are those corresponding to $|t| \lesssim \beta$. The effective-mass approximation is therefore inadequate whenever the average motion of the impurity through the surrounding fluid over an interval β is significantly different from that of a free particle. Using a relaxation time derived from the experimental mobility and estimated values of the effective mass,[4] it can be seen that a better approximation for F is necessary for both positive and negative ions in He^3 whenever the temperature is appreciably less than 0.3°K. The effects of such a modification will now be considered.

In the situations of interest the effect of the surrounding fluid is in general to reduce the average distance traveled by the impurity in a time of order β. This has the effect of increasing the real part of F, thus making the mobility less than that expected from the effective-mass approximation. In the extreme case where the distance moved by the impurity in time β is small compared with an inverse Fermi wave vector, $F(K, t)$ is approximately unity for $|t| \lesssim \beta$, and μ_s becomes equal to $(\sigma n k_F)^{-1}$, where $\sigma = 2\pi\int_0^\pi d\theta \times \sin\theta(1-\cos\theta)\sigma(\theta)$ is the momentum-transfer scattering cross section. This situation appears to be a fair approximation to reality, the observed mobilities at low temperatures being greater than $(\sigma n k_F)^{-1}$ (taking for σ the geometrical momentum-transfer cross section πR^2) by factors in the range 1.5-2.0.

Egelstaff and Schofield[8] have shown that the terms of order K^2 in $F(K, t)$ are expressible in terms of the frequency spectrum of the motion of the particle. The latter can be related in turn to the frequency-dependent mobility $\mu(\omega)$ by using the fluctuation-dissipation theorem.[9] This is sufficient to evaluate F in a Gaussian approximation:

$$F(K, t) = \exp\{-\tfrac{1}{2}K^2\gamma(t)\}, \quad (3)$$

where

$$\gamma(t) = \frac{2}{\pi}\int_0^\infty d\omega \, \mathrm{Re}\,\mu(\omega) \frac{\cosh\frac{1}{2}\beta\omega - \cos\omega(t - \frac{1}{2}i\beta)}{\omega \sinh\frac{1}{2}\beta\omega}. \quad (4)$$

The limiting form of (4),

$$\gamma(t) = \frac{2\mu_s}{\pi}\left[\ln\sinh\frac{\pi t}{\beta} + \ln\frac{\beta\omega_c}{\pi} + C\right] - i\mu_s \,\mathrm{sgn}\, t, \quad (5)$$

where C is Euler's constant and ω_c is defined in

such a way as to make

$$\lim_{\Omega\to 0}\int_{\Omega}^{\infty} d\omega \frac{\mathrm{Re}\,\mu(\omega)}{\omega} + \mu_s \ln\frac{\Omega}{\omega_c}$$

vanish, is applicable whenever both t and β are large compared with the inverse of the lowest frequency at which $\mu(\omega)$ departs significantly from μ_s. On the assumption that this frequency is $(\mu_s M)^{-1}$ (which is $\approx 10^{11}$ sec^{-1} for both positive and negative ions), (5) is a good approximation to use in (2) at temperatures below about 0.1°K, provided that a suitable cutoff is applied to remove the divergence at small t.

If $\mu(\omega)$ can be considered to be characterized by the single relaxation time $\mu_s M$, then Eqs. (2)-(4) form a closed set to be solved self-consistently for the static mobility μ_s. The actual frequency dependence of μ is more complicated. Nevertheless, these equations can be used to estimate an upper limit for the temperature variation of μ_s for the negative ions. Because μ_s is known experimentally to be very weakly temperature dependent, for fixed τ most of the temperature dependence of $\gamma(\beta\tau)$ comes from the explicit β dependence of the term $(2\mu_s/\pi)\ln(\beta\omega_c/\pi)$ in (5). Hence $(\partial/\partial \ln\beta)F(K,\beta\tau)\approx -(\mu_s K^2/\pi)F(K,\beta\tau)$ and so by (2),

$$(\partial/\partial \ln\beta)\mu_s^{-1}\approx -\langle K^2\rangle/\pi, \qquad (6)$$

where $\langle K^2\rangle$ is a weighted average of K^2 [actually the mean square momentum transfer per collision, weighted by a factor $(1-\cos\theta)$]. $\langle K^2\rangle$ is expected to lie between zero and $(8/3)k_F^2$, which is the weighted mean for isotropic scattering. The predicted mobility exponent $\alpha = \partial \ln\mu_s/\partial \ln T$ is thus negative, and its absolute magnitude is at most $(8/3\pi)\mu_s k_F^2$.

For negative ions, $-\alpha$ is predicted to be less than 0.1, in contrast to the previous predictions giving a value ≈ 2 at temperatures below T_0. In practice, it appears that in the region where the experiments have been carried out, terms of order T^2, neglected in the above, dominate the temperature dependence and change its sign. However, at sufficiently low temperatures the T^2 terms should be unimportant, and the mobility temperature coefficient should then be negative. At still lower temperatures (of the order of microdegrees), a number of assumptions used in deriving (6) break down, and the ultimate low-temperature behavior of the mobility may well be T^{-2} in agreement with Refs. 1-3. For positive ions the assumptions are already appreciably violated in the 30- to 100-mdeg region. It is clear, however, that the present theory may give a basis for understanding the comparatively weak temperature dependence observed.

In summary, we have shown that the effective-mass approximation requires modification at low temperatures, and that better agreement with experiment can be obtained by describing the dynamics of the ions in terms of the Van Hove scattering function.

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