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EFFECT OF ELECTRON TRAPPING ON THE ION-WAVE INSTABILITY*

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It is found that in many cases electron trapping can have an important effect on ion waves. Trapping can render the linear stability theory invalid, and the conditions under which this may occur are discussed. Assuming the trapping process to be the mechanism whereby an unstable plasma reaches a quasistationary turbulent state, we also derive an expression for the saturation energy spectrum.

In this Letter we discuss the effect of electron trapping on the ion-wave instability in a plasma, the instability arising because of the presence of an electron current. Two fundamental questions are raised: One is whether the stability criterion derived from the linear theory could be modified by trapping; the other is whether, even when linear stability theory does remain valid, the trapping process can act as a mechanism which brings an unstable plasma to a quasistationary turbulent state.

All linear theories[1-7] of ion-wave instability implicitly assume that electron trapping is unimportant. This assumption is reasonable only if we insist that the waves have "very small" amplitude. However, in reality there is a natural lower bound to the amplitude. Fluctuations associated with the thermally generated ion oscillations provide a lower bound for the energy of an ion wave and so for its amplitude. Restricting ourselves to thermal ion-wave fluctuations, we can estimate the bounce frequency of the trapped electrons in a typical potential trough, the trough having a certain characteristic width of the order of the electron Debye length, say. The electron may be said to be trapped only when the bounce period is much shorter than the lifetime of the potential trough (or more precisely, the correlation time of the fluctuating fields of a certain wave packet). This time is at least of the order of some typical ion-wave period, say ω_s^{-1}. Evidently when the bounce frequency ω_b greatly exceeds the wave frequency ω_s the linear stability theory can be in error, unless some other mechanisms, such as collisions, suppress the trapping process. In fact it is interesting, as we show later, that the Coulomb collisions can efficiently prevent trapping from taking place.

We now examine the ratio ω_b/ω_s in order to find under what conditions it exceeds unity. Since the electron Debye length, λ_e, represents a typical length scale for the ion oscillations, we are interested in the most probable potential depth, φ, with width of the order of this characteristic length. Because, in general, the fluctuation fields consist of a large number of propagating modes, the potential φ may be estimated in some average sense. Thus, using the autocorrelation function, we write

$$\varphi^2 = |\langle \varphi(\vec{r})\varphi(\vec{r}+\vec{\lambda}_e)\rangle|, \tag{1}$$

where $\langle\rangle$ denotes the ensemble average. Assuming isotropic fluctuation and using the random phase approximation, i.e.,

$$\langle \varphi_{\vec{k}}\varphi_{-\vec{k}'}\rangle = (2\pi)^3\delta[\vec{k}-\vec{k}']I_k, \tag{2}$$

where φ_k is the Fourier transform of $\varphi(r)$, we can rewrite (1) as

$$\phi^2 = \frac{1}{2\pi^2}\left|\int_0^{k_{\max}} dk\, k^2 \frac{\sin k\lambda_e}{k\lambda_e} I_k\right|. \tag{3}$$

Since the wave energy per ion-wave mode U_k can be expressed as[8]

$$U_k = \frac{k^2 I_k}{8\pi}\left[\frac{\partial}{\partial\omega}\{\omega\epsilon(\vec{k},\omega)\}\right]_{\omega=\omega_k}, \quad (4)$$

where $\epsilon(k,\omega)$ is the dielectric function and ω_k the ion-wave frequency, expression (3) as a function of U_k takes the form

$$\phi^2 = \frac{2}{\pi\lambda_e}\left|\int_0^{k_{\max}} dk\,\frac{kU_k \sin k\lambda_e}{k^2+k_e^2}\right|. \quad (5)$$

In deriving (5) we have used the approximate result

$$\left[\frac{\partial}{\partial\omega}\{\omega\epsilon(\vec{k},\omega)\}\right]_{\omega=\omega_k} \simeq 2\left(1+\frac{k_e^2}{k^2}\right), \quad (6)$$

for $k^2 < k_e^2$, with $k_e^2 = \lambda_e^{-2}$. Though relation (6) is accurate only in the region $k^2 < k_e^2$ it suffices for an order-of-magnitude analysis. Choosing $k_{\max} \simeq k_e$ and remembering that[4]

$$U_k = T_e, \quad k \lesssim k_e,$$
$$= 0, \quad k > k_e, \quad (7)$$

we find from (5)

$$\phi^2 \simeq T_e/\lambda_e. \quad (8)$$

If we now define the trapping velocity v_t and bounce frequency ω_b as

$$v_t \equiv (2e\phi/m)^{1/2} \text{ and } \omega_b \equiv v_t/\lambda_e$$

we obtain

$$\frac{\omega_b}{\omega_s} \simeq \frac{v_t}{v_s} = \left(\frac{M}{m}\frac{2e\phi}{T_e}\right)^{1/2} \simeq \left(\frac{M}{m}\right)^{1/2}(n\lambda_e^3)^{-1/4}, \quad (9)$$

where m and M are the electron and ion mass, and $v_s = (T_e/M)^{1/2}$. Thus if we define the small parameter $g = (n\lambda_e^3)^{-1/4}$, it is meaningful to speak of electron trapping when

$$g > (m/M)^{1/2}, \quad (10)$$

a condition frequently met by many laboratory plasmas.

We point out that condition (10), by itself, does not insure that electron trapping takes place. The reason is that in the preceding discussion we have neglected the effect of Coulomb collisions which may, in practice, be very efficient in suppressing the trapping. While we discuss this point in more detail in a forthcoming paper, here we state the conclusion. First of all, let ν be a collision frequency which may be expressed in terms of the parameter g as $\nu = \alpha\omega_e|g^4 \ln g|$ (α is a coefficient of order unity and ω_e is the electron plasma frequency); then when in the quasi-one-dimensional case the condition

$$\omega_b \simeq (v_e/v_t)^2\nu \quad (11)$$

is satisfied, the effects of trapping and collisions become competitive. We now want to examine the validity of the linear stability theory for a current-carrying plasma. For this purpose we should note that in such a plasma the long-wavelength component of the ion-wave fluctuation is strongly enhanced even when the plasma is approaching instability. For such a plasma, we should choose the characteristic length of the ion-wave potential, l say, to be greater than λ_e. Taking this into account, we find that condition (11) can now be written in terms of g as

$$|g_c \ln g_c| \simeq \chi(k_e l), \quad (12)$$

where $\chi(k_e l)$ is a function which is of order unity for $l \sim k_e^{-1}$, decreasing as l increases, and where g_c represents a certain "critical" value of g, defined so that when g is much smaller than g_c collisions cannot destroy trapping.

We may now draw two conclusions: First, if the plasma is in a state such that

$$g_c \gg g \gg (m/M)^{1/2}, \quad (13)$$

trapping is important and the instability predicted by the linear theory may not even occur since trapping can "quench" Landau growth. Second, if the parameter g is such that $g > (m/M)^{1/2}$ and $g \gg g_c$, collisions suppress trapping by thermal fluctuations and the usual ion-wave instability may exist. However, once the fluctuations have grown to a certain level, trapping may set in and serve as a "saturation" mechanism and determine the quasistationary turbulent energy spectrum. This leads us to a new theory of ion-wave turbulence quite different from the previous theories,[9-15] which favor the quasilinear and mode-coupling interactions. In our opinion, however, there are many cases where trapping can be more important than these mechanisms.

Returning to our discussion, we consider the case $g > (m/M)^{1/2}$ and $g > g_c$ and remark, as we demonstrate in a forthcoming publication, that then the usual quasilinear process is, in fact, unimportant or even meaningless. Restricting ourselves now to the trapping regime, we see that the saturation level of the wave energy is determined by the condition

$$\nu(v_e/\tilde{v}_t)^2 \simeq \tilde{\omega}_b. \quad (14)$$

The difference between (11) and (14) is that in (11) v_t and ω_b are estimated for fluctuations at the thermal level and in (14) $\tilde{v}_t$ and $\tilde{\omega}_b$ are defined for the saturation level. Since λ_e no longer represents a typical length scale, we should consider $\tilde{v}_t = v_t(l)$ and $\tilde{\omega}_b = \omega_b(l)$. We note that $\tilde{v}_t$ is proportional to $\phi^{1/2}$ and $\omega_b(l) = v_t(l)/l$. Thus from the saturation condition (14), we see that ϕ must be proportional to $l^{2/3}$, since $\tilde{v}_t^2\tilde{\omega}_b \simeq V_e^2\nu$ which is independent of l. Furthermore, since the very long-wavelength modes of the fluctuations cannot grow (collisional damping prevails over the weak Landau growth), we expect the saturation potential $\phi(l)$ to drop very rapidly when l exceeds some upper-bound value, say l_0. Hence we speculate that $\phi(l)$ behaves like

$$\phi(l) = Al^{2/3}\exp(-l/l_0), \tag{15}$$

where A is a coefficient independent of l. In order to determine the energy spectrum we should also introduce an angular cutoff in the k integral since only those modes with wave vector nearly parallel to the electron current can grow. In the following we introduce this cutoff by simply writing

$$\langle|\varphi_{\vec{k}}|^2\rangle = \langle|\varphi_k|^2\rangle \text{ for } 0 \le \theta \le \theta_0(k) \ll 1,$$
$$= 0 \text{ for } \theta_0(k) < \theta. \tag{16}$$

Noting the relation

$$\varphi_l^2 = \left|\int \frac{d\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{l}} \langle|\varphi_{\vec{k}}|^2\rangle\right|$$
$$\simeq \int_0^\infty \frac{dk}{8\pi^2} k^2\theta_0^2(k)\langle|\varphi_k|^2\rangle \cos kl \tag{17}$$

we find by Fourier inversion

$$\langle|\varphi_k|^2\rangle = \frac{16\pi}{k^2\theta_0^2}\int_0^\infty dl\phi^2(l)\cos kl. \tag{18}$$

Substituting (15) into (18) we obtain for $l_0^{-2} \ll k^2 \ll k_e^2$

$$U_{\vec{k}} \simeq \frac{k_e^2}{4\pi}\langle|\varphi_{\vec{k}}|^2\rangle \propto \frac{k^{-13/3}}{\theta_0^2(k)}. \tag{19}$$

To obtain the frequency spectrum, U_ω, we integrate over all possible directions of $\vec{k}$ for fixed k and get

$$U_\omega \sim \omega^{-7/3}.$$

From this we see that the wave energy piles up in the low-frequency region. Physically, this is what we expect, since the longer the wavelength, the longer is the time required for a complete trapped-particle oscillation and hence the greater is the possibility that trapping is disrupted by a collision. Thus, the longer the wavelength, the greater is the saturation value of the wave amplitude.

One can also evaluate the maximum amplitude of the wave from (14). We find

$$\frac{U_k}{T_e} \simeq \frac{1}{\theta_0^2}(k_e l_0)^{13/3}(g \ln g)^{4/3}.$$

From this we see that the saturation level is lower the smaller the parameter g provided $g > (m/M)^{1/2}$. This is in contrast to what obtains for saturation by the mode-coupling process.

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