BREAKDOWN OF ASYMPTOTIC SUM RULES IN PERTURBATION THEORY

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It is shown that all of the principal results of the Bjorken-limit technique break down in perturbation theory in the "gluon" model of strong interactions.

Three years ago Bjorken[1] pointed out that the asymptotic behavior of a time-ordered product of two currents is related to equal-time commutators of the currents and their time derivatives,

$$\lim_{\substack{q_0\to i\infty \\ \vec{q}\text{ fixed}}} \int d^4x\, e^{-iq\cdot x} T(J_\mu{}^a(x) J_\nu{}^b(0)) = (iq_0)^{-1}\int d^4x\, e^{-iq\cdot x}\delta(x^0)[J_\mu{}^a(x), J_\nu{}^b(0)] + (iq_0)^{-2}\int d^4x\, e^{-iq\cdot x}\delta(x^0)[\dot{J}_\mu{}^a(x), J_\nu{}^b(0)] + O(q_0{}^{-3}), \quad (1)$$

$$\dot{J}_\mu{}^a(x) = (\partial/\partial x^0) J_\mu{}^a(x).$$

This connection has been extensively applied to the study of radiative corrections to hadronic β decay[2] and to the derivation of asymptotic sum rules[3] and asymptotic cross-section relations[4] for high energy inelastic electron and neutrino scattering. In all of these applications, it is assumed that the equal-time commutators appearing on the right-hand side of Eq. (1) are the same as the "naive commutators" obtained by straightforward use of canonical commutation relations and equations of motion. That this is a questionable assumption was pointed out by Johnson and Low,[5] who independently discovered Eq. (1). They studied this equation in a simple perturbation-theory model, in which the currents couple through a fermion triangle loop to a scalar (vector) meson. They found that in most cases the results obtained by explicit evaluation of the left-hand side of Eq. (1) differ from those calculated from naive commutators by well-defined extra terms. Because of special features of the triangle graph model, however, these extra terms do not directly invalidate the applications of Eq. (1) mentioned above.

We report here the results of a more realistic perturbation theory calculation, which shows that for commutators of space components with space components, the Bjorken limit and the naive commutator differ by terms which modify all of the principal applications of Eq. (1). We consider a simple, renormalizable model of strong interactions, consisting of an SU(3) triplet of spin-$\frac{1}{2}$ particles ψ bound by the exchange of an SU(3)-singlet massive vector "gluon." The vector current in this model is $J_\mu{}^a = \bar{\psi}\gamma_\mu\lambda^a\psi$, and the naive equal-time commutator of two vector currents is

$$\delta(x^0-y^0)[J_\mu{}^a(x), J_\nu{}^b(y)] = \delta^4(x-y)\bar{\psi}(x)C\psi(x), \quad (2)$$

$$C = \tfrac{1}{2}\{\lambda^a,\lambda^b\}(\gamma_\mu\gamma_0\gamma_\nu - \gamma_\nu\gamma_0\gamma_\mu) + \tfrac{1}{2}[\lambda^a,\lambda^b](\gamma_\mu\gamma_0\gamma_\nu + \gamma_\nu\gamma_0\gamma_\mu).$$

We wish to compare the Bjorken-limit commutator with the naive commutator, to second order in the gluon-fermion coupling constant g, in the special case in which Eqs. (1) and (2) are sandwiched between fermion states. To do this, we calculate the renormalized current-fermion scattering amplitude $\tilde{T}_{\mu\nu}{}^{ab}(p,p',q)$ and compare it, in the limit as $q_0\to i\infty$, with the renormalized vertex $\tilde{\Gamma}(C;p,p')$ of the naive commutator.[6] The scattering amplitude can be expressed in terms of the renormalized vector vertex $\tilde{\Gamma}(\gamma_\mu;p,p')$ and the renormalized fermion propagator $\tilde{S}(p)$ by

$$\tilde{T}_{\mu\nu}{}^{ab}(p,p',q) = \tilde{\Gamma}(\gamma_\mu;p,p+q)\tilde{S}(p+q)\tilde{\Gamma}(\gamma_\nu;p+q,p')\lambda^a\lambda^b + \tilde{\Gamma}(\gamma_\nu;p,p-q')\tilde{S}(p-q')\tilde{\Gamma}(\gamma_\mu;p-q',p')\lambda^b\lambda^a + B_{\mu\nu}{}^{ab}(p,p',q), \quad (3)$$

with $B_{\mu\nu}{}^{ab}(p,p',q)$ the sum of the two box diagrams illustrated in Fig. 1. We find, by explicit calculation,

$$\lim_{\substack{q_0\to i\infty \\ \vec{q},\, p,\, p' \text{ fixed}}} \tilde{T}_{\mu\nu}{}^{ab}(p,p',q) = q_0{}^{-1}[\tilde{\Gamma}(C;p,p')+\Delta] + O(q_0{}^{-2}\ln q_0), \tag{4}$$

$$\Delta = (g^2/16\pi^2)\{2(g_{\mu\nu}-g_{\mu 0}g_{\nu 0})\gamma_0[\lambda^a,\lambda^b] + \tfrac{3}{2}(\gamma_\nu\gamma_0\gamma_\mu - \gamma_\mu\gamma_0\gamma_\nu)\{\lambda^a,\lambda^b\}\}.$$

We see that the Bjorken-limit commutator and the naive commutator differ by the term labeled Δ, which is well defined and finite. We note that Δ vanishes when $\mu=0$ or $\nu=0$, indicating that for the time-time and time-space commutators, the Bjorken limit and the naive commutator agree. This result can be independently deduced from the usual on-shell Ward identity

$$q^\mu \tilde{T}_{\mu\nu}{}^{ab}(p,p',q) = \tilde{\Gamma}([\lambda^a,\lambda^b]\gamma_\nu;p,p'); \tag{5}$$

the consistency between Eq. (4) and Eq. (5) provides a convenient check on the calculation leading to Eq. (4). When one or both currents $J_\mu{}^a$, $J_\nu{}^b$ is replaced by the corresponding axial-vector current $J_\mu{}^{5a} = \bar{\psi}\gamma_\mu\gamma_5\lambda^a\psi$, $J_\nu{}^{5b}$, a formula like Eq. (4) holds, with the appropriate change in C and with Δ modified as follows:

$$J_\mu{}^a \to J_\mu{}^{5a} \Longleftrightarrow \Delta \to -\gamma_5\Delta,\quad J_\nu{}^b \to J_\nu{}^{5b} \Longleftrightarrow \Delta\to\Delta\gamma_5,\quad J_\mu{}^a\to J_\mu{}^{5a}, J_\nu{}^b \to J_\nu{}^{5b} \Longleftrightarrow \Delta \to -\gamma_5\Delta\gamma_5 = \Delta. \tag{6}$$

One may wonder whether our definition of $\tilde{\Gamma}(C;p,p')$ could be changed by a finite rescaling in such a way as to absorb the term Δ. However, since $\gamma_\mu\gamma_0\gamma_\nu + \gamma_\nu\gamma_0\gamma_\mu \propto g_{\mu 0}\gamma_\nu + g_{\nu 0}\gamma_\mu - g_{\mu\nu}\gamma_0$ and since $\gamma_\mu\gamma_0\gamma_\nu - \gamma_\nu\gamma_0\gamma_\mu \propto \epsilon_{\mu 0\nu\lambda}\gamma^\lambda\gamma_5$, the vertex $\tilde{\Gamma}(C;p,p')$ is a linear combination of vector and axial-vector vertices. Therefore, the normalization of this vertex is completely fixed by the time-component current algebra and Lorentz covariance, and rescaling is not permitted.

In addition to studying the $q_0{}^{-1}$ term in Eq. (1), we have also calculated the $q_0{}^{-2}$ term in the special case considered by Callan and Gross.[4] Specializing to forward scattering ($p=p'$, $a=b$) and spin averaging, we find

$$\lim_{p_0\to\infty}\lim_{q_0\to i\infty} m p_0{}^{-2} q_0{}^2 \tfrac{1}{4}\mathrm{tr}\left[\left(\frac{\gamma\cdot p+m}{2m}\right)\tilde{T}_{ij}{}^{aa}(p,p,q)\right]$$
$$= -2(\delta^{ij}-\hat{p}^i\hat{p}^j)(\lambda^a)^2 + \frac{g^2}{6\pi^2}[2(\ln q_0{}^2 + \mathrm{const})(\delta^{ij}-\hat{p}^i\hat{p}^j) + \hat{p}^i\hat{p}^j](\lambda^a)^2. \tag{7}$$

The presence of $\ln q_0{}^2$ on the right-hand side of Eq. (7) indicates that the expression of Eq. (1) cannot, strictly speaking, be carried out to order $q_0{}^{-2}$, and that the coefficient $\langle p|\delta(x^0)[\dot{J}_i{}^a(x), J_j{}^a(0)]|p\rangle$ of the $q_0{}^{-2}$ term is logarithmically divergent.[7] Using naive commutators to evaluate this coefficient, Callan and Gross concluded that the double limit on the left-hand side of Eq. (7) should be proportional to the transverse tensor $\delta^{ij}-\hat{p}^i\hat{p}^j$. The presence of the additional term $(g^2/6\pi^2)\hat{p}^i\hat{p}^j(\lambda^a)^2$ in Eq. (7) indicates that their conclusion fails in perturbation theory.

We next indicate how the various applications of the Bjorken-limit technique are modified by our results.

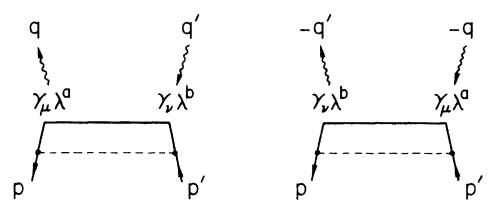

FIG. 1. Box diagrams contributing to $B_{\mu\nu}{}^{ab}$. The dashed line denotes the virtual gluon.

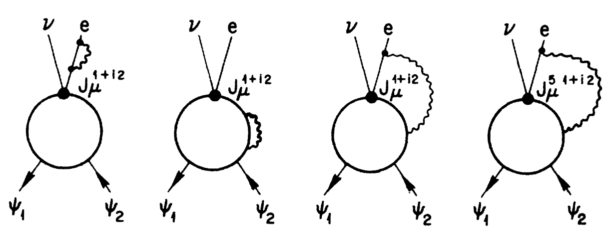

FIG. 2. Diagrams for the radiative corrections to the vector β transition. The wavy line denotes the virtual photon.

(i) Radiative corrections to β decay.[2]—We consider the vector β transition between the fermions ψ_1 and ψ_2. We introduce a cutoff Λ^2 and calculate the divergent part of the radiative corrections to this process, described by the diagrams of Fig. 2. Using the time-component current algebra alone, it has been shown that the first three diagrams in Fig. 2 sum to a universal, structure-independent fractional change in the decay amplitude $\delta M/M = (3\alpha/8\pi)\ln\Lambda^2$. The divergent part of the fourth diagram in Fig. 2 can be evaluated to order g^2 from Eqs. (4) and (6), giving $\delta M/M = (3\alpha/8\pi)2\overline{Q}\ln\Lambda^2(1-3g^2/16\pi^2)$, with $\overline{Q}$ the average charge of the doublet $\psi_{1,2}$. The term proportional to g^2 comes, of course, from the isospin-symmetric term in Δ. The total divergent part of the radiative correction is thus, to order g^2,

$$(\delta M/M)_{\rm total} = (3\alpha/8\pi)\ln\Lambda^2[1+2\overline{Q}(1-3g^2/16\pi^2)]. \tag{8}$$

We see that the choice $\overline{Q} = -\frac{1}{2}$, which removes the divergence to lowest order in g^2, still leaves a residual divergence in second order.

(ii) Asymptotic sum rules and cross-section relations.[3,4]—We introduce the variable $\omega = -q^2/p\cdot q$ and define the spectral functions $W_1(\omega,q^2)$ and $W_2(\omega,q^2)$ by

$$\frac{\mathrm{disc}_\omega}{-2\pi i}\frac{1}{4}\mathrm{tr}\left[\left(\frac{\gamma\cdot p+m}{2m}\right)\tilde{T}_{\mu\nu}{}^{ab}(p,p,q)\right] = \lambda^a\lambda^b[W_1(\omega,q^2)(-g_{\mu\nu}+q_\mu q_\nu/q^2)$$
$$+W_2(\omega,q^2)(p_\mu - p\cdot q q_\mu/q^2)(p_\nu - p\cdot q q_\nu/q^2)],\quad \omega>0. \tag{9}$$

In terms of these spectral functions, the asymptotic formula of Eq. (7) may be rewritten as the sum rule

$$\lim_{q^2\to-\infty}(\cdots)(\delta^{ij}-\hat{p}^i\hat{p}^j)+\frac{g^2}{6\pi^2}\hat{p}^i\hat{p}^j = \lim_{q^2\to-\infty} 2\int_0^2 \omega d\omega\,[mW_1(\delta^{ij}-\hat{p}^i\hat{p}^j)+(mW_1+q^2mW_2/\omega^2)\hat{p}^i\hat{p}^j], \tag{10}$$

first obtained by Callan and Gross. As these authors note, the quantity $mW_1+q^2mW_2/\omega^2$ is positive definite, differing only by positive factors from $-q^2\sigma_L(\omega,q^2)$, with σ_L the longitudinal electroproduction cross section. Thus the presence of $\hat{p}^i\hat{p}^j$ in Eq. (7) implies that, in the quark model, $q^2\sigma_L(\omega,q^2)$ does not vanish asymptotically, in disagreement with the conclusion of Callan and Gross. In a similar fashion, the SU(3)-antisymmetric part of Eq. (4) leads to the asymptotic sum rule

$$1-\frac{g^2}{8\pi^2} = \lim_{q^2\to-\infty} -2\int_0^2 d\omega\, mW_1. \tag{11}$$

Apart from the term $g^2/8\pi^2$ on the left-hand side, which comes from the SU(3)-antisymmetric part of Δ, Eq. (11) is the backward-neutrino-scattering asymptotic sum rule of Bjorken.[3] The modification in the left-hand side of Eq. (11) is closely related to the nonvanishing of $q^2\sigma_L(\omega,q^2)$. To see this, we write down the usual fixed-q^2, time-component algebra sum rule[8]

$$1 = 2\int_0^2 d\omega\, q^2 mW_2/\omega^2 \tag{12}$$

and subtract it from Eq. (11), giving[9]

$$-g^2/8\pi^2 = \lim_{q^2\to-\infty} -2\int_0^2 d\omega(mW_1+q^2mW_2/\omega^2). \tag{13}$$

Thus the SU(3)-antisymmetric term in Δ and the $\hat{p}^i\hat{p}^j$ term in Eq. (7) are basically the same phenomenon. As an additional check on our arithmetic, we have calculated W_1 and W_2 directly, giving $mW_1 + q^2mW_2/\omega^2 = g^2\omega/32\pi^2$, in agreement with Eqs. (10) and (13).

We have also studied the scalar (pseudoscalar) gluon model in perturbation theory, and find effects similar to those reported here. Full details of the calculations, and further discussion, will be published elsewhere.

We wish to thank W. A. Bardeen and S. B. Treiman for helpful discussions, and Dr. Carl Kaysen for the hospitality of the Institute for Advanced Study. After this work was completed, we learned that R. Jackiw and G. Preparata had also discovered the breakdown of the Callan-Gross result in perturbation theory.

Note added in proof. – (i) We have been informed by J. D. Bjorken of a related paper by A. I. Vainshtein and B. L. Ioffe {Zh. Eksperim. i Teor. Fiz.–Pis'ma Redakt. 6, 917 (1967) [translation: Soviet Phys.–JETP Letters 6, 341 (1967)]}. We will discuss this work in our detailed paper. (ii) In the case when one current is an axial-vector current [the first two lines of Eq. (6)], we have omitted an SU(3)-singlet contribution to the Bjorken limit coming from the triangle diagram discussed by Johnson and Low. Addition of this piece does not alter any of our conclusions.

[1]J. D. Bjorken, Phys. Rev. 148, 1467 (1966).

[2]For references, see G. Preparata and W. I. Weisberger, Phys. Rev. 175, 1965 (1968).

[3]For a survey, see lectures by J. D. Bjorken, in Selected Topics in Particle Physics, Proceedings of the International School of Physics "Enrico Fermi," Course XLI, edited by J. Steinberger (Academic Press, Inc., New York, 1968).

[4]C. G. Callan and D. J. Gross, Phys. Rev. Letters 22, 156 (1969).

[5]K. Johnson and F. E. Low, Progr. Theoret. Phys. (Kyoto) Suppl. Nos. 37-38, 74 (1966).

[6]The renormalized vertex $\tilde{\Gamma}(C;p,p')$ is obtained from the unrenormalized vertex $\Gamma(C;p,p')$ by multiplying by the fermion wave-function renormalization constant Z_2, with no further finite rescalings.

[7]The term $\ln q_0^2$ is present when p_0 is finite and is not a result of the additional $p_0 \to \infty$ limit.

[8]See S. L. Adler and R. F. Dashen, Current Algebras (W. A. Benjamin, Inc., New York, 1968), Chap. 4.

[9]This connection was first noted by F. J. Gilman, Phys. Rev. 167, 1365 (1968).

VECTOR-MESON DOMINANCE AND HIGH-ENERGY ELECTRON-PROTON INELASTIC SCATTERING*

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A vector-meson dominance model for high-energy ep inelastic scattering is proposed. We predict that at high values of the missing mass (>2.5 BeV) the ratio of the longitudinal to the transverse cross section is given by $\xi(q^2/m_\rho^2)[1-(q^2/2m_p\nu)]^2$, where ξ is expected to be of the order of unity. Scale invariance (in the sense of Bjorken) is satisfied at very high q^2. The model may adequately account for the remarkably large cross section recently observed in the deep inelastic region.

The conjecture that the total (hadronic) photoabsorption cross section may be calculated from diffractive photoproduction of[1] ρ, ω, and φ has been brilliantly confirmed by recent experiments.[2] This implies that the hypothesis of vector-meson dominance can successfully account for virtual Compton scattering on protons of transversely polarized, $q^2<0$ (timelike in our metric) photons. It is generally conceded, however, that vector-meson dominance fails for spacelike values of q^2 because the photoabsorption cross section for spacelike photons deduced from electron-proton inelastic scattering[3] falls off with q^2 far more slowly than is expected from a $[m_\rho^2/(m_\rho^2+q^2)]^2$-type formula.[4] In this Letter we show that, when the longitudinal contribution is properly calculated, the hypothesis of vector-meson dominance is completely compatible with existing experimental data that do not separate the longitudinal from the transverse cross section. Our model makes a number of specific predictions; apart from its simplicity, its main virtue is that it can be readily destroyed by experiment.

We start by writing out the ep inelastic differential cross section in terms of W_1, W_2, σ_T, and σ_S:

$$\frac{d^2\sigma}{dq^2d\nu}=\frac{E'}{E}\frac{4\pi\alpha^2}{q^4}\left[W_2(q^2,\nu)\cos^2\frac{\theta}{2}+2W_1(q^2,\nu)\sin^2\frac{\theta}{2}\right], \tag{1}$$

$$W_2=\frac{K}{4\pi^2\alpha}\frac{q^2}{q^2+\nu^2}(\sigma_T+\sigma_S),\quad W_1=\frac{K}{4\pi^2\alpha}\sigma_T,\quad E-E'=\nu,\quad q=(\vec{q},i\nu)=\text{virtual-photon momentum},$$

$$K=\nu-(q^2/2m_p)=(s-m_p^2)/2m_p,$$

where $\sqrt{s}$ = missing (hadronic) mass. The transverse and longitudinal cross sections σ_T and σ_S (in