PROBES FOR THE CONSTITUENTS OF THE ELECTROMAGNETIC CURRENT AND ANOMALOUS COMMUTATORS*

R. Jackiw† and G. Preparata‡
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 24 February 1969)

It is shown that the Callan-Gross sum rules for electroproduction fail in perturbation theory. Nonperturbative lepton annihilation sum rules are derived. The inapplicability of perturbation theory is traced to the breakdown of the Johnson-Low-Bjorken high-energy theorem.

The recent work of Bjorken[1] and Callan and Gross[2] (CG) concerning the behavior of total cross sections for electroproduction off protons at high energies had indicated the feasibility of determining the constituents of the electromagnetic current in terms of basic fields. CG find that if the current is bilinear in quark fields, the longitudinal cross section satisfies

$$\lim_{-q^2\to\infty} q^2\sigma_L = 0, \tag{1}$$

where q^2 is the momentum transfer. In the above limit $\omega = -q^2/\nu$ is held fixed, where $\nu = p\cdot q$ and p is the proton momentum. On the other hand, if the current is bilinear in spinless boson fields or proportional to a vector field, then the transverse cross section σ_T vanishes asymptotically at fixed ω,

$$\lim_{-q^2\to\infty} q^2\sigma_T = 0. \tag{2}$$

These results follow from the tensor structure of the object

$$C_{ij}(\vec{p}) = \int d^3x \langle p|[\dot{J}_i(\vec{x},0), J_j(0)]|p\rangle, \tag{3}$$

where the equal-time commutator arises from an application of the Johnson, Low,[3] and Bjorken[4] (JLB) high-energy theorem to the time-ordered product of two currents.

In the case of the quark model, CG determine the tensor structure of (3) by appeal to equations of motion and equal-time commutation relations for the unrenormalized quark fields out of which the current is constructed. They assert that for scalar, pseudoscalar, or vector interactions $C_{ij}(\vec{p})$ has the form

$$C_{ij}(\vec{p}) = (\delta_{ij}\vec{p}^2 - p_ip_j)A + \delta_{ij}B, \tag{4}$$

where A and B are Lorentz scalars, hence independent of $\vec{p}^2$.

We have checked the CG result (1) in perturbation theory as well as in a Bethe-Salpeter summation for the above models. It is found that (1) does not hold. The reason for this has been traced to the fact that the JLB high-energy theorem is not satisfied in perturbation theory. In particular the tensor structure (4), as determined by formal manipulation with unrenormalized fields, does not coincide with the tensor structure of the high-energy limit of the appropriate T product. This in turn is a consequence of the divergences of the unrenormalized theory. The structure for $C_{ij}(\vec{p})$, which we obtain by explicitly calculating $\int dx\, e^{iq\cdot x}\langle p|T[J_i(x), J_j(0)]|p\rangle$ and taking the high-energy limit, is of the form (4), except A is logarithmically divergent, while B is finite and proportional to $\vec{p}^2$. Thus the longitudinal part of $C_{ij}(\vec{p})$ is not the space-space component of a second-rank Lorentz tensor, but rather involves space-space-time-time components of a fourth-rank tensor. This then has the consequence that (1) does not hold. The detailed calculations will be presented elsewhere.[5]

It must be concluded therefore that the CG sum rule (1) is a result about unrenormalized field theory, not verifiable in perturbation theory. It is seen that it possesses the same degree of reliability as the old sum rules for the high-energy behavior of unrenormalized propagators, for the unrenormalized mass, etc.

If one is willing to accept nonperturbative results based on unrenormalized field theory, a simple sum rule for total electron-positron annihilation may be derived. This sum rule also provides a test of the constituents of the electromagnetic current. Furthermore it predicts that asymptotically the ratio of the total cross section for annihilation into leptons is determined solely by the charge structure of the hadronic and leptonic fermion fields.

To derive this, we define

$$\sum_n \langle 0|J_\mu(0)|n\rangle\langle n|J_\nu(0)|0\rangle(2\pi)^4\delta^4(p_n-q) = (q_\mu q_\nu - g_{\mu\nu}q^2)\rho(q^2). \tag{5}$$

The total cross section σ_h (σ_l) for the process $e^+ + e^- \to$ hadrons (leptons) at center-of-mass energy $(q^2)^{1/2}$ is given by[6]

$$\sigma_{h,l}(q^2) = 16\pi^2\alpha^2\rho_{h,l}(q^2)/q^2. \tag{6}$$

The subscripts h, l, on ρ indicate that the hadronic and leptonic currents, respectively, are to be taken in (5).

In terms of the above spectral function, the Schwinger term is given by

$$\langle 0|[J_0(\vec{x},0), J_i(0)]|0\rangle = -i\partial_i\delta^3(\vec{x})\frac{1}{2\pi}\int_0^\infty dm^2\rho(m^2), \tag{7a}$$

or

$$\int_0^\infty dm^2\rho(m^2) = -i\frac{2\pi}{3}\int d^3x\, x^i\langle 0|[J_0(\vec{x},0), J_i(0)]|0\rangle. \tag{7b}$$

In the algebra of fields, the Schwinger term is finite; thus we conclude that

$$\lim_{+q^2\to\infty} q^4\sigma_h(q^2) = 0 \quad \text{(algebra of fields).} \tag{8}$$

To calculate the Schwinger term in quark models, we define $J_i(0)$ to be $\bar{\psi}(\frac{1}{2}\epsilon)\gamma_i Q\psi(-\frac{1}{2}\epsilon)$, where Q is the charge matrix, and ϵ is a spacelike vector, to be set to zero symmetrically at the end of the calculation. Using the unrenormalized commutators

$$[J_0(\vec{x},0), \psi(\vec{y},0)] = -Q\psi(\vec{x},0)\delta^3(\vec{x}-\vec{y}) \tag{9a}$$

we find that

$$\langle 0|[J_0(\vec{x},0), J_i(0)]|0\rangle = -[\delta(\vec{x}+\tfrac{1}{2}\vec{\epsilon}) - \delta(\vec{x}-\tfrac{1}{2}\vec{\epsilon})]\langle 0|\bar{\psi}(\tfrac{1}{2}\epsilon)\gamma_i Q^2\psi(-\tfrac{1}{2}\epsilon)|0\rangle, \tag{9b}$$

or from (7b)

$$\int_0^\infty dm^2\rho(m^2) = -i(2\pi/3)\epsilon^i\langle 0|\bar{\psi}(\epsilon)\gamma_i Q^2\psi(0)|0\rangle. \tag{10}$$

To complete the calculation, we must obtain an expression for the right-hand side of (10). This will contain terms which diverge as $\epsilon\to 0$, as well as terms that are finite as $\epsilon\to 0$. We concentrate on the former, and show that they are independent of the interactions of the fermion fields. To calculate the divergent portion of

$$\lim_{\epsilon\to 0}\epsilon^i\langle 0|\bar{\psi}(\epsilon)\gamma_i Q^2\psi(0)|0\rangle,$$

we now give ϵ a positive time component and consider

$$\lim_{\epsilon\to 0}\epsilon^i\langle 0|T[\bar{\psi}(\epsilon)\gamma_i Q^2\psi(0)]|0\rangle = \lim_{\epsilon\to 0}\epsilon^i \operatorname{Tr}[\gamma_i Q^2 G(\epsilon)], \tag{11}$$

where $G(x)$ is the unrenormalized fermion propagator,

$$G(\epsilon) = \langle 0|T[\psi(0)\bar{\psi}(\epsilon)]|0\rangle = \int\frac{d^4p}{(2\pi)^4}e^{-ip\epsilon}g(p). \tag{12}$$

As $\epsilon\to 0$, $G(\epsilon)$ has a singularity in ϵ, determined by the large-p behavior of $g(p)$. Since for large p

$$g(p) = -\frac{i}{\not{p}} + O\left(\frac{1}{p^2}\right), \quad G(\epsilon) = -\frac{i}{2\pi^2}\frac{\not{\epsilon}}{\epsilon^4} + O\left(\frac{1}{\epsilon^2}\right). \tag{13}$$

The term of $O(1/\epsilon^2)$ gives a contribution to $\epsilon^i G(\epsilon)$ which is odd in ϵ, hence vanishes upon averaging over ϵ; while terms of $O(1/\epsilon)$ give finite contributions of no interest to us in this calculation. The result important in the present application is that the coefficient of $\not{\epsilon}/\epsilon^4$ which determines the divergent part of $\int_0^\infty dm^2\rho(m^2)$ is independent of interactions. If we split $\rho(m^2)$ into $\rho^0 + \tilde{\rho}$, where $\tilde{\rho}$ is the contribution to ρ from the interactions, we see that $\int_0^\infty dm^2\rho^0(m^2)$ diverges while $\int_0^\infty\tilde{\rho}(m^2)dm^2$

converges. Assuming that $\tilde{\rho}(m^2)$ possesses no oscillatory pathologies, we conclude that $\rho^0(m^2)$ will dominate $\tilde{\rho}(m^2)$ for large m^2. Thus

$$\lim_{q^2\to\infty}\frac{\rho_h(q^2)}{\rho_l(q^2)}=\lim_{q^2\to\infty}\frac{\rho_h{}^0(q^2)}{\rho_l{}^0(q^2)}=\frac{\mathrm{Tr}Q_h{}^2}{\mathrm{Tr}Q_l{}^2}=\mathrm{Tr}Q_h{}^2. \tag{14}$$

In the last equality we have used the fact the lepton charge matrix, Q_l, is unity.

The above argument makes use of the classic result that the unrenormalized propagator tends to the free propagator at large momenta. This is in fact an example of the JLB limit, which states in the present application that

$$g(p)=\int dx\, e^{ipx}\langle 0|T[\psi(0)\bar{\psi}(x)]|0\rangle \underset{p_0\to\infty}{\to} -\frac{i}{p_0}\int d^3x\, e^{-i\vec{p}\cdot\vec{x}}\langle 0|\{\psi(0),\bar{\psi}(\vec{x},0)\}|0\rangle+O\left(\frac{1}{p_0{}^2}\right). \tag{15a}$$

Use of the canonical field commutators gives

$$g(p)\to -i\gamma^0p^0/p_0{}^2+O(1/p_0{}^2). \tag{15b}$$

Since $g(p)$ is covariant, and a function only of p, the above must covariantize into

$$g(p)\to -i/\not{p}+O(1/p^2),$$

which coincides with (13).

Finally we may calculate ρ^0 explicitly. This is given by[7]

$$\rho_{h,l}{}^0(q^2)=\frac{1}{12\pi^2}\mathrm{Tr}Q_{h,l}{}^2+O\left(\frac{1}{q^2}\right); \tag{16}$$

therefore in contrast to (8) we find

$$\lim_{q^2\to\infty}\frac{q^2\sigma_{h,l}(q^2)}{16\pi^2\alpha^2}=\frac{1}{12\pi^2}\mathrm{Tr}Q_{h,l}{}^2 \quad \text{(quark model)}. \tag{17}$$

Thus Eqs. (6) and (7) offer a test between the algebra of fields and the quark model. Furthermore, if the quark-type model is satisfied, then Eq. (14) or (17) determines the charge structure of the quarks.

The present quark-model calculations cannot be verified in perturbation theory. The reason is obvious: The unrenormalized propagator does not exist, and Eq. (13) cannot be given meaning in perturbation theory. Furthermore[8] the result that only ρ^0 contributes to the divergent part of $\int_0^\infty dm^2\rho(m^2)$ would also imply that only this quantity causes $\int_0^\infty (dm^2/m^2)\rho(m^2)$ to diverge. However the latter object is just $Z_3{}^{-1}$, and in perturbation theory Z_3 diverges in every order.

As the only input has been the application of the JLB limit, (15a), we conclude that this does not hold in perturbation theory for $g(p)$. This is a trivial breakdown, obviously traceable to the nonexistence of $\langle 0|T[\psi(0)\bar{\psi}(x)]|0\rangle$. However our discovery that the JLB limit is also inapplicable to $\langle p|T[J_\mu(0)J_\nu(x)]|p\rangle$ is remarkable since that object exists in (renormalized) perturbation theory. Thus all applications of that limit in perturbation theory are suspect and one must check in each individual case whether or not the limit is applicable. We suspect that when the formal commutator of well-defined operators yields finite matrix elements, then the JLB limit is valid. However when infinite (cutoff dependent) or ambiguous matrix elements result from the formal commutator, then the JLB limit cannot be applied with impunity.

Although our quark model sum rules, as well as those of Callan and Gross, are based on unrenormalized field theory, we do not believe that they are of formal interest only. They can be checked experimentally and their verification would be a hint that unrenormalized field theory converges, in spite of the evidence of perturbative calculations.

It is a pleasure to acknowledge conversations with Professor C. Callan, Professor S. Glashow, and Professor K. Johnson.

Note added in proof. — Upon completion of this investigation, we learned from Dr. S. Adler that he too has discovered the breakdown of the JLB limit in the CG as well as other applications.

*Work supported in part by the Office of Naval Research, U. S. Navy, Contract No. Nonr-1866(55).

†Junior Fellow, Society of Fellows.

‡Fulbright Scholar.

[1]J. D. Bjorken, Phys. Rev. (to be published).

[2]C. G. Callan and D. J. Gross, Phys. Rev. Letters 22, 156 (1969).

[3]K. Johnson and F. E. Low, Progr. Theoret. Phys. (Kyoto) Suppl. Nos. 37-38, 74 (1966).

[4]J. D. Bjorken, Phys. Rev. 148, 1467 (1966).

[5]R. Jackiw and G. Preparata, to be published.

[6]Bjorken, Ref. 4, Sec. X. Speculations about the problem we consider are to be found in this work.

[7]N. N. Bogoliubov and D. V. Shirkov, Introduction to the Theory of Quantized Fields (Interscience Publishers, Inc., New York, 1959).

[8]We are grateful to Professor K. Johnson for emphasizing this point for us.